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On the Convergence, Consistence and Stability Analysis for New Numerical Procedure for the Solution of $y' = f(x,y)\ y(a) = y_o$

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Abstract: Omowaye proposed a new numerical procedure for the solution of initial value problem in ODEs. The proposed scheme is particularly suited to initial value problems having oscillatory and exponential solutions. In this study, we revise their formulae and extend the work to include among other things, perturbation of the interpolating function, a complete and compact discussion of the convergence, consistence and stability of the new scheme.

Key words: Oscillatory and exponential solution, convergence, consistence, stability, interpolant

INTRODUCTION

Let us consider the initial value problem (ivp)

$$y' = f(x,y),\ y(a) = y_O \qquad (1)$$

$y, f \varepsilon \Re^m$ and $x \varepsilon [a,b]$ a finite interval on the real line.

The mathematical formulation of many problems in applied simulation, astronuautics, nuclear-reactor theory, control, engineering and medicine often leads to the initial value problems (ivps).

It is a well known fact that mathematical techniques now play an important role in planning and implementation of government policies. Also, it must be remarked here that the efficiency of any method depends on the method's stability and certain accuracy properties. The accuracy properties of different methods are usually compared by considering the order of convergence as well as truncation errors (Ibijola and Kama, 1999).

THE NUMERICAL PROCEDURES REVISED

We shall follow Owom and modified the trigonometrical term in the interpolating function and make the following assumptions that the theoretical solution y(x) to Eq. 1 can be locally represented in the interval $[x_n, x_{n+1}]$ by the perturbation of a polynomial interpolating function with a trigonometric and exponential function. This general interpolating function is of the form

$$F(x) = p_m(x) + b\sin(Nx + A) + Ce^x \qquad (2)$$

The polynomial $p_m(x)$ is given as

$$p_m(x) = \sum_{j=0}^{m} a_j x^j \quad m \geq 0 \qquad (3)$$

Where the unknown coefficient (real) a_j ,b and c are undermined coefficient and N,A are real oscillatory parameters. For the purpose of this study we shall take the simplest form of (2). This implies that we shall take the case of m = 0 in (3)then

$$F(x) = a_0 + b\sin(Nx + A) + C\,e^x \qquad (4)$$

At the point x_n and x_{n+1} (4) becomes

$$F(x_n) = a_0 + b\sin(Nx_n + A) + Ce^{x_n} = y_n \qquad (5)$$

$$F(x_{n+1}) = a_0 + b\sin(Nx_{n+1} + A) + Ce^{x_{n+1}} = y_{n+1} \qquad (6)$$

We proceed further to add one more assumption. Let $f^{(i)}$ denote the i^{th} total derivative of f(x,y) with respect to x, we assume that

$$F^{(1)}(x_n) = f(x_n, y_n) = f_n \qquad (7)$$

$$F^{(2)}(x_n) = f'(x_n, y_n) = f_n' \qquad (8)$$

$$F'(x_n) = N\ \ b\cos(Nx_n + A) + Ce^{x_n} = f_n \qquad (9)$$

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$$F^{(2)}(x_n) = -N^2 b\sin(Nx_n + A) + Ce^{x_n} = f_n' \quad (10)$$

We now use Eq. 9 and 10 to determine b and c completely as

$$b = -\left[\frac{f_n' - f_n}{N\cos(Nx_n + A) + N^2 \operatorname{Sin}(Nx_n + A)}\right] \quad (11)$$

$$C = \frac{N^2 \sin(Nx_n + A) f_n + N\cos(Nx_n + A) f_n'}{(N\cos(Nx_n + A) + N^2 \sin(Nx_n + A))e^{x_n}} \quad (12)$$

Equation 6 minus (5) will lead to a new one-step method.

$$y_{n+1} - y_n = \left[\sin(Nx_{n+1} + A) - \sin(Nx_n + A)\right] b + \left[e^{x_{n+1}} - e^{x_n}\right]C \quad (13)$$

If we set $\theta(x_n) = Nx_n + A = \theta_n$, $x_{n+1} - x_n = h$, we obtain

$\theta_{n+1} = Nx_{n+1} + A = Nx_n + A + Nh = \theta_n + Nh$

First and foremost, we expand $\sin\theta_{n+1} = \sin(\theta_n + Nh)$

$$\sin(\theta_n + Nh) = \sin\theta_n \cos Nh + \sin Nh \cos\theta_n \quad (14)$$

We observed that,

$$\cos Nh = 1 - \frac{(Nh)^2}{2|} + \frac{(Nh)^4}{4|} - \frac{(Nh)^6}{6|} + \ldots.$$

$$\sin Nh = Nh - -\frac{(Nh)^3}{3!} + \frac{(Nh)^5}{5!} + \ldots\ldots\ldots\ldots$$

$$\text{So, } \sin(\theta_n + Nh) - \sin\theta_n = \left[-\frac{N^2h^2\sin\theta_n}{2!} + \frac{N^4h^4\sin\theta_n}{4!} - \frac{N^6h^6\sin\theta_n}{6!} + \ldots.\right] + \left[Nh\cos\theta_n - \frac{N^3h^3\cos\theta_n}{3!} + \frac{N^5h^5\cos\theta_n}{5!} + \ldots\ldots\right] \quad (15)$$

$$e^{x_{n+1}} - e^{x_n} = e^{x_{n+h}} - e^{x_n} = e^{x_n}(e^h - 1)$$

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\ldots\ldots \quad (16)$$

$$e^{x_{n+1}} - e^{x_n} = e^{x_n}\left(h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots.\right)$$

Substituting (11), (12), (15) (16) in (13) we have

$$y_{n+1} - y_n = h\left[(M + E)f_n' + (D - M)f_n\right] \quad (17)$$

$$\text{where } D = \frac{\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\ldots.\right)N\sin\theta_n}{\cos\theta_n + N\sin\theta_n},$$

$$E = \frac{\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \ldots.\right)N\cos\theta_n}{\cos\theta_n + N\sin\theta_n}$$

$$M = -\left[\frac{Nh\sin\theta_n}{2!} + \frac{N^3h^3\sin\theta_n}{4!} - \frac{N^5h^5\sin\theta_n}{6!} + \ldots.. + \cos\theta_n - \frac{N^2h^2\cos\theta_n}{3!} + \ldots.\right] \Big/ (\cos\theta_n + N\sin\theta_n)$$

CONVERGENCE ,CONSISTENCE AND STABILITY OF THE ALGORITHM

In this study, we show that the algorithm (17) is not only convergent but stable and consistent.

Consistency: The concept of consistency of any one-step method is very important in the sense that it control the magnitude of the local truncation error and it is crucial to the convergence of one-step method.

Definition 1: Fatunla (1988) A numerical scheme with an incremental function $\phi(x_n, y_n; h)$ is said to be consistent with the initial value problem (1) if,

$$\phi(x_n, y_n; 0) = f(x_n, y_n) \quad (18)$$

A consistent method has order of at least one. If we put h = 0 in (17) after substituting the value of M,D,E we have Eq. 18 which confirm that the method is consistent.

Convergence: We shall discuss the convergence of this one-step method by putting it in form of a theorem:

Theorem: A one- step numerical integrator of the form is convergent if and only it is consistent.

$$y_{n+1} - y_n = h\phi(x_n, y_n; h)$$

Proof: From Eq. 17 we have

$$y_{n+1} - y_n = h\,\phi_n \quad (19a)$$

$$\text{Where } \phi_n = (M + E)f_n' + (D - M)f_n \quad (19b)$$

Substituting the value of M,D and E into (19b) leading to

$$\phi_n |_{h=0} = \frac{\cos\phi_n}{\cos\phi_n + N\sin\phi_n} f_n' + \frac{N\sin\phi}{\cos\phi_n + N\sin\phi_n} f_n \quad (20)$$

$$\phi_{n!h=0} = p\ f_n' + R\ f_n \quad (21)$$

where $p = \frac{\cos\phi_n}{\cos\phi_n + N\ \mathrm{Sin}\ \phi_n}$ and $R = \frac{n\sin\phi_n}{\cos\phi_n + N\sin\phi_n}$

By Fatunla (1988) and Lambert (1976) the integrator is consistent with the initial value problem and hence the integrator is convergent.

Stability: The stability of the method is equally tested by applying the formula (17) to solve the Dahlquist stability scalar test initial value problem

$$y' = \lambda y, y(x_0) = y_0 \quad (22)$$

Under the assumption that Re (λ)<0 and that the equation has a steady state solution

$$y(x) = y(x_0)e^{\lambda x} \quad (23)$$

This leads to the first order difference equation

$$y_{n+1} = \phi(z) y_n \quad (24)$$

Where the stability function is obtained as

$$\phi(z) = 1 + Qz + s\lambda \quad (25)$$

where $Q = (M + E)$, $S = D - M$ with $Z = h\lambda$
for simplicity and easy evaluation of (25) we assume Q=s= 1, then (25) becomes

$$Q(z) = 1 + z + \lambda \quad (26)$$

Obviously, the recurrent relation (24) provides a semi-A-stable solution using definition 2,3,4 of Omowdye (2006) provided

$$|Q(z)| < 1 \quad (27)$$

CONCLUSION

In this study, a full review of Omowaye *et al*. (2005) has been presented and extension of the research to include among other things is a compact but detail analysis of the convergence ,consistency and stability of the method has been given.

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