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On the Superiority of Chebyshev Polynomials to Bessel Polynomial in Solving Ordinary Differential Equation

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Abstract: In this study, we show the superiority of Chebyshev polynomials to Bessel polynomials in solving first order ordinary differential equation with rational coefficients. Shifted Chebyshev polynomials, Bessel polynomials as well as Canonical polynomial were generated in solving the differential equation of our choice in order to show the superiority of Chebyshev polynomials to Bessel polynomial. Numerical examples are given which show the superiority of Chebyshev polynomials to Bessel polynomials.

Key words: Chebyshev polynomials, Bessel polynomials, canonical polynomials hypergeometric series, perturbation

INTRODUCTION

Lanczos (1938) first proposed the accurate approximate polynomial solutions of Linear Ordinary differential equations with polynomial coefficients via Tau method. Techniques based on this method have been used in many literatures notable among them are Ortiz (1969), Onumanyi (1962), Freilich and Ortiz (1982) while technique based on Chebyshev have been discussed by Fox (1962, 1980) and Onamanyi (1962).

In this study, canonical polynomial is constructed based on the associated conditions of the given problem with a simple recursive relation via Chebyshev and Bessel polynomials.

The study of Ortiz (1969) gives an account of the theory of the Tau method which is subsequently, used on the problems considered to illustrate the effectiveness and superiority of Chebyshev polynomials to Bessel polynomials.

DESCRIPTION OF THE METHOD USED

Following Lanczos (1938, 1974) we define canonical polynomials Q_k (x), $k\geq 0$ which are uniquely associated with the operator. Canonical polynomial offers several advantages: Canonical polynomial neither depend on the boundary conditions of the problem which we want to solve nor on the interval in which the solution is sought and they are easily generated using a single recursive relations.

Consider a linear differential equation y'-4y =0, y (0) =1 which defined the exponential function

$$y = e^{4x} \tag{1}$$

Generating the canonical polynomials

$$L = \frac{d}{dx}. \qquad y = x^k$$
$$Lx^k = kx^{k-1} - 4x^k$$
$$\text{Thus; } Lx^k - kLQ_{k-1}(x) + 4LQ(x) = 0$$

From the linearity of L and the existence of L^{-1}, we have

$$x^k = 4Q(x) = kQ_{k-1}(x)$$
$$\text{Since } DQ(x) = x^k$$

From the boundary conditions

$$\begin{gathered} Dy(x) = 0 \; x^k = 0 \\ 4Q_k(x) = kQ_{k-1}(x) \\ Q_k(x) = \tfrac{1}{4}Q_{k-1}(x) \end{gathered} \tag{2}$$

It follows that

$$Q_k(x) = \tfrac{1}{4}k!S_k$$

Chebyshev polynomials: Following Fox (1962), Onumanyi (1962) and George (1974) we recall some well

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known properties of the chebyshev polynomial of degree n Where this leads to the recursive relation

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x) \quad (3)$$

Where n =1, 2, 3,
This will be for our subsequent use

$$\text{For } T_0(x) = T_0(\cos\theta) = 1$$
$$T_1(x) = T_1(\cos\theta) = x, \text{ e.t.c.}$$

Shifted Chebyshev polynomials: By choosing shifted chebyshev polynomials $T^*_n(x) = x$ define with the help of the linear transformation

Let $x \in [-1,1] \Rightarrow \theta \in [0, 1]$ sup pose $\theta = \alpha x + \beta$ using the transformation, we have

However, to determine the shifted Chebyshev polynomial

$$T^*_n(x) = \sum_{k=0}^{n} C^n_k x^k = \sum_{k=0}^{n} C^n_k \theta^k$$

We shall use the chebyshev polynomial to get the shifted chebyshev polynomial by putting x = 2θ -1 and to determine the values of the C^n_k we equate $T^*_n(x)$ to $T_n(x)$

Bessel polynomial: Bessel polynomial $J^*_{2n}(x)$ was introduce by Bessel Friedrich Wilhelm (1784-1846) a German Astronomer and Mathematician, defined the hypergeometric series.

$$J^*_{2n}(x) = F\left(1, \tfrac{1}{4}, n+1; -x\right) = F(\alpha, \beta, \gamma; -x)$$

$$\text{Where } J^*_{2n}(x) = \sum_{k=0}^{n} J^{2n}_{2k} x^{2k}$$

then when n = 1,we have

$$J^*_2(x) = J^2_0 + J^2_2 x = 8 - x^2$$
$$J^2_0 = 8,\ J^2_2 = -1$$

When n=2 e. t. c

$$J^*_4(x) = J^4_0 - J^4_2 x^2 + J^4_2 x^4 = 96 - 8x^2 + 5x^4$$
$$J^4_0 = 96,\ J^4_2 = -8,\ J^4_2 = 5$$

The Tau method: Ortiz (1969) gives an account of the theory of Tau Method, which is applied to the following basic problem.

$$Ly(x) = P_m(x)y^m(x) + \ldots + P_0 y(x) = F(x) \text{ for } a \le x \le b$$

$y^m(x)$ stands for the derivative of order m of y (x) and $y(x) = y_n(x)$

$$y(x) = y_n(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_i Q_i(x)$$

Where $Q_i(x)$ is the Canonical polynomial. Here we need a small perturb term which leads to the choice of Chebyshev polynomial which oscillates with equal amplitude in the range considered, $P_m(x) = \tau T^*_n(x)$ where $T^*_n(x)$ is the shifted chebyshev polynomials which are often used with the Tau method and

$$T_n(x) = \sum_{k=0}^{n} C^n_k x^k$$

Where C^n_k are coefficient of x^k we assume here that a transformation in 2 had be used to get the shifted chebyshev polynomial.

NUMERICAL EXAMPLE 1

Consider the differential equation

$$y' - 4y = 0, \qquad y(0) = 1 \quad (4)$$

This defines the exponential function

$$y(x) = e^{4x} = 1 + 4x + \frac{16}{2!}x^2 + \frac{64}{3!}x^3 + \ldots \quad (5)$$

This converges in the entire complex plain. If we truncate the series, we have

$$y(x) = 1 + 4x + \frac{16}{2!}x^2 + \frac{64}{3!}x^3 + \ldots + \frac{4^n}{n!}x^n$$

This function satisfies the differential equation

$$y_n' - y_n(x) = \frac{4^n}{n!}x^n \quad (7)$$

Suppose we are solving (4) in the range [0, 1]. Now by choosing Chebyshev polynomial T*n (x) defined with the help of the hypergeometric series
$T^*n(x) = F(-n, n, \tfrac{1}{2}; x)$ as the error term on the right had side of (4). We therefore perturbed and solve the differential equation.

$$y_n' - 4y_n = \tau T_n(x) y_n' - 4y_n = \tau T_n(x) \quad (8)$$

By introducing an auxiliary set of polynomials called Canonical polynomial $Q_k(x)$, in which $Q_k(x)$ is constructed as shown in Eq. 10

$$\text{i.e. } Q_k(x) = -\tfrac{1}{4} k! S_k$$

If we denote its partial sum of the first (k+1) terms of the series by $S_k(x)$ such that

$$S_k(x) = 1 + 4x + \frac{16}{2!}x^2 + \frac{64}{3!}x^3 + \ldots + \frac{4^n}{n!}x^n \quad (9)$$

Writing out polynomial $T_n(x)$ explicitly as

$$T_n(x) = \sum_{k=0}^{n} C_k^n x^k \quad (10)$$

By superposition of linear operator we have

$$y_n(x) = -\tfrac{1}{4}\tau \sum_{k=0}^{n} C_k^n k!\, S_k(x) \quad (11)$$

Satisfy the boundary condition $y_n(0) = 1$, will yields

$$-\tfrac{1}{4}\tau \sum_{k=0}^{n} C_k^n k!\, S_k(0) = 1$$

$$-\tau = \frac{4}{\sum_{k=0}^{n} C_k^n k!}$$

The final solution for the nth approximate becomes

$$y_n(x) \frac{-\sum_{k=0}^{n} C_k^n k!\, S_k(x)}{\sum_{k=0}^{n} C_k^n k!} = \frac{M_n(x)}{R_n(x)} \quad (12)$$

Where $M_n(x) = \sum_{k=0}^{n} C_k^n k! S_k(x)$, and $R_n(x) = \sum_{k=0}^{n} C_k^n k!$

when n = 8,
Recall that

$$T_8(x) \sum_{k=0}^{n} C_k^n x^k$$

$$1 - 128x + 2688x^2 + 21504x^3 + 884480x^4 + 180224x^5 + 212992x^6 + 131072x^7 + 327668x^8 \; R_8(x) = 794233985.$$

$$M_8(x) = S_0(x) - 128S_1(x) + 5376S_2(x) - 129024S_3(x) + 207520S_4(x) - 21626880S_5(x) + 153354240S_6(x) - 660602880S_7(x) + 1221205760S_8(x)$$

$$S_8(x) = 1 + 4x + \frac{16}{2!}x^2 + \frac{64}{3!}x^3 + \frac{256}{4!}x^4 + \frac{1024}{5!}x^5 + \frac{4396}{6!}x^6 + \frac{16384}{7!}x^7 + \frac{68536}{8!}x^8$$

$$M_8(x) = 794233985 + 3176935936x + 6353872880x^2 + 847177383x^3 + 84731442x^4 + 6761218047x^5 + 46305116617x^6 + 2147483647x^7 + 2147483648x^8$$

$$y_n(x) = \frac{M_n(x)}{R_n(x)}$$

$$y_8(1) = \frac{M_8(1)}{R_8(1)} = \frac{4.29566403 \times 10^{10}}{794233985}$$

$$= 54.08566599 \quad (13)$$

Against the true value $y_1(1) = 54.59815003$ (14)

Thus, error = 0.51248408 (15)

However, the unweighted partial sum $S_8(1)$ gives 53.4311746603 with an error of 1.166404.

Here, we see the great increased convergence been obtained. However, assuming the range [0, 1] is accidental. If we further dealing with analytic functions, which is defined at all points of the complex plane except for singular point. Hence our aim will be to obtain y(z) where z may be chosen our error polynomial in the term $T_n^*(x/z)$ and solve the given differential equation along the complex ray which connects the point $x = 0$ with the point $x = z$. Then solving the differential equation.

$$Dy_n(x) = \tau T_n^*(x/z) \quad (16)$$

By considering z merely as a given constant we finally substitute for x, the end point x = z.

Now, we replace the coefficient of the Chebyshev polynomial by the corresponding coefficients of the Bessel polynomials $J^*_n(x)$ defined by the hypergeometric series.

$$J^*_n(x) = F(1, 1/4, n+1, -x^2)$$

Then we obtain

$$y_n^j(x) \frac{\sum_{k=0}^{n} J_k^n k! \frac{S_k(x)}{z^k}}{\sum_{k=0}^{n} \frac{J_k^n k!}{z^k}} = \frac{B_n(x)}{T_n(x)} \quad (17)$$

Where

$$B_n(z) = \sum_{k=0}^{n} J_k^n k! \frac{S_k(x)}{z^k}$$

$$T_n(z) = \sum_{k=0}^{n} \frac{J_k^n k!}{z^k}$$

The previous approximations have now turned into rational approximations giving the successive approximates as the ratio of two polynomials.

When n = 8, we have

$$y_8(z) = \frac{B_8(z)}{T_8(z)}$$

Where

$$y_8(z) = \frac{B_8(z)}{T_8(z)}$$

$$T_8(z) = 107520z^8 + 107520z^6 + 53760z^4 + 1555200z^2 + 707616600 \text{and} B_8(z) = 124630016z^8 + 2243918848z^7 + 419584512z^6 + 62063160z^5 + 767285760z^4 + 761011200z^3 + 567648000z^2 + 283046400z + 70761600$$

Hence

$$y_8^j(z) = \frac{B_8(z)}{T_8(z)} \text{ is obtained}$$

Putting z = 1, we have

$$y_8^*(z) = \frac{3857522496}{7248832}$$

With an error of 1.368950593 compare with the previous error of 0.51248408. This gives much closer e-values than the values obtained by the Chebyshev weighting. Hence, the discrepancy becomes more pronounced as we go to approximation of increasing order.

Table 1 shows for some numerical results for the estimates based on the example given when x = 1.

Table 1: Numerical results for the estimates based on the example given when x = 1

	Chebyshev polynomial $y^c(1)$		Bessel polynomial $y^b(1)$	
N	Approximate	Error	Approximate	Error
6	51.17487459	3.42327544	50.82585752	3.77229251
8	54.42696612	0.51249868	53.451267667	1.14688206
10	54.5477885	0.05237118	54.4484807	0.14970196

CONCLUSION

The polynomials of Chebyshev and Bessel have been discussed. The result obtained in the present research shows the effectiveness and superiority of Chebyshev polynomials to Bessel polynomials for the solution of first order linear differential equation. The Bessel polynomials fail to give better values than the Chebyshev polynomials even at the end point x = 1.

However, the condition that our domain shall contain no singular points is satisfied. Therefore, Bessel polynomials $J^*_n(x)$ gives larger errors than the Chebyshev polynomials $T^*_n(x)$ for increasing values of n.

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