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The Components of Stress on an Internally Pressurized Hollow Circular Cylinder of the Blatz-Ko Material

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Abstract: The problem of determining explicit formulas for a hollow cylinder under uniform internal pressure is the thrust of this study. The cylinder under consideration is made of the Blatz-Ko material †. In obtaining the required results, the radial displacement of the cylinder is sought by means of (an approximate) series solution of an emerging boundary-value problem.

Key words: Components, stress, internally, hollow, circular cylinder

INTRODUCTION

A substantial amount of research was done by Chung *et al*. (1986) in investigating the solutions of the problems of the Blatz-ko materials. The result of their investigation in the aspect of a hollow circular cylinder is the parametric representation of the radial displacement of the cylinder as well as its components of stress. Away from such parametric representation, Ejike and Erumaka (2006) provided an alternative approach to the solution of a similar problem. Their approach achieved a transformation of an emerging non-linear boundary-value problem to a form in which the non-linear boundary conditions are amenable to linearization. This done, a solution may be sought via any genuine mathematical procedure. Unfortunately, as much as this writer knows, at present working mathematical formulas are still needed for the computation of the components of stress on a hollow circular cylinder under applied uniform internal pressure. This research seeks to address this problem.

In this study we shall formulate a boundary-value problem for the axi-symmetric deformation of the hollow cylinder whose mapping is given by

$$R_0 \to R : (r,\theta) \to (R,\Theta) \qquad (1)$$

where (R, Θ) is the polar coordinate of the deformed region. The solution to the boundary-value problem posed in this study shall be determined. The solution (i.e., the determination of the radial displacement R_0 (r) of the cylinder) shall be sought by means of (an approximate) series solution of the non-linear boundary-value problem.

FORMULATION OF BOUNDARY-VALUE PROBLEM

We now consider the hollow circular cylinder under an applied uniform internal pressure of magnitude P. In its underformed configuration, let its cross-section be defined by the region.

$$R_0 = [(r,\Theta); r \in (a,b), \theta \in (0,2\pi)\} \qquad (2)$$

where α is the inner radius and is the outer radius. The deformed region is given by the mapping

$$R_0 \to R : (r,\theta) \to (R,\Theta) \qquad (3)$$

Where (R_0, Θ) is the polar coordinate of the deformed region. Supposing that the deformation is an axi-symmetric plain strain one, then we have

$$\left.\begin{aligned} &R = R(r) > 0, R(r) \in C^2(a,b) \\ &\Theta = \theta \text{ on } R_0 \end{aligned}\right\} \qquad (4)$$

Let F be the deformation gradient tensor associated with (4). with this we have:

$$F = \begin{pmatrix} \dot{R} & 0 \\ 0 & \frac{R}{r} \end{pmatrix} \qquad (5)$$

where R = dR/dr . The right Cauchy-Green deformation gradient tensor B is, in the present case,

$$B = F^T F = \begin{pmatrix} \dot{R}^2 & 0 \\ 0 & \left(\frac{R}{r}\right)^2 \end{pmatrix} a \qquad (6)$$

The tensor B is so defined in terms of the strain energy function W (I,J), whose invariants I and J are

$$\left.\begin{aligned} I &= spB = \dot{R} + \left(\frac{R}{r}\right)^2 \\ J &= (\det B)^{1/2} = \frac{\dot{R}R}{r} \end{aligned}\right\} \qquad (7)$$

We find that the principal stretches associated with (3) are

$$\lambda_r = \dot{R}(r), \lambda_\theta = \frac{R(r)}{r} \qquad (8)$$

A class compressible isotropic elastic materials (including the Blatz-ko) is characterized by the elastic potential (Chung *et al.*, 1986; Ejike and Erumaka, 2006)

$$W(I,J) = \frac{\mu}{2}\left(\frac{I}{J^2} + 2J - 4\right) \qquad (9)$$

Where $\mu > 0$ is the shear modulus of the materials at infinitesimal deformations. If σ is the stress tensor associated with the plane deformation, then tensor σ is given by

$$\sigma = \frac{2\phi B}{J} + \psi I \, , \qquad (10)$$

where the response functions ϕ (I, J) and ψ (I, J) are defined in terms of W (I, J) and are given by

$$\phi = \frac{\partial W}{\partial I}, \psi = \frac{\partial W}{\partial J} \qquad (11)$$

Consequently, we have

$$\phi = \frac{\mu r^2}{2\dot{R}^2 R^2}, \psi = \mu\left(1 - \frac{r^3 \dot{R}^2 + rR^2}{\dot{R}^3 R^3}\right) \qquad (12)$$

Substituting (6) and (12) in (10) we get

$$\sigma = \mu\left[\begin{pmatrix} \frac{r^3}{\dot{R}R^3} & 0 \\ 0 & \frac{r}{\dot{R}^3 R} \end{pmatrix} + \begin{pmatrix} 1 - \frac{r^3 \dot{R}^2 + rR^2}{\dot{R}^3 R^3} & 0 \\ 0 & 1 - \frac{r^3 \dot{R}^2 + rR^2}{\dot{R}^3 R^3} \end{pmatrix}\right] \qquad (13)$$

hence the components of stress

$$\sigma_{RR}(r) = \mu\left(1 - \frac{r}{\dot{R}^3 R}\right)$$

$$\sigma_{\theta\theta}(r) = \mu\left(1 - \frac{r^3}{\dot{R}R^3}\right)$$

$$\sigma_{R\theta} = \sigma_{\theta R} = 0, \; r \in (\alpha, b) \qquad (14)$$

Neglecting all body forces, we have the equation of equilibrium as †Hill and Arrigo (1996) showed that equilibrium equations for perfectly elastic compressible matriels can be obtained as Lagrange equations for the variationals principles

$$\delta\left\{ \iint_{B_0} \sum (I, J, \lambda) R dR dz \right\} = 0$$

where
the arguments of the strain energy function Σare defined

$$\frac{d\sigma_{RR}}{dr} + \frac{\lambda_r(\sigma_{RR} - \sigma_{\theta\theta})}{r\lambda_\theta} = 0, r \in (a,b)$$

i.e.

$$\frac{d\sigma_{RR}}{dr} + \frac{\dot{R}(\sigma_{RR} - \sigma_{\theta\theta})}{R} = 0, r \in (a,b) \qquad (15)$$

Equation (15) together with (14) yield the non-linear ordinary differential equation for R (r):

$$3rR^3 \ddot{R} - R^3 \dot{R} + r^3 \dot{R}^4 = 0, r \in (a,b) \qquad (16)$$

with the boundary conditions

$$\sigma_{RR} = -p \text{ at } r = a, \sigma_{RR} = 0 \text{ at } r = b \qquad (17)$$

In using (14), we may rewrite (17) as

$$R(a)\dot{R}^3(a) = a\left(1 + \frac{p}{\mu}\right)^{-1}, R(b)\dot{R}^3(b) = b \qquad (18)$$

In what follows we shall derive a solution to the boundary-value problem (16), (18).

SOLUTION OF BOUNDARY-VALUE PROBLEM

The first task here is to linearize the boundary conditions (18). Recently, this method was applied by Ejike and Erumaka (2006) in solving the problem of deformation of a rotating circular cylinder. Motivated by this, we let

$$R^4(r) = S^3(r) \tag{19}$$

With this, (16) becomes†

$$\left(3r\ddot{S} - \dot{S}\right)S^3 - \frac{3}{4}rS^2\dot{S}^2 + \left(\frac{3}{4}\right)^3 r^3\dot{S}^4 = 0 \tag{20}$$

and (18) yields

$$\dot{S}(b) = \frac{4}{3}\left(\frac{\mu a}{\mu + p}\right)^{\frac{1}{3}}, \dot{S}(b) = \frac{4}{3}b^{\frac{1}{3}} \tag{21}$$

where

$$\dot{S} = \frac{ds}{dr}$$

We now seek a series solution, for S

(r), of the form

$$S(r) = a_1 r^m + a_2 r^{m-1} + a_2 r^{m-2} + \ldots, a_1 \neq 0, \tag{22}$$

where $a_1, a_2, a_3, \ldots$ and the index m are constants to be determined. We obtain the necessary derivatives of S (r) from (22) according as the requirements of (20) and substitute accordingly, then we equate the coefficients of the powers of r to zero. The result was shown in Nzerem (2006) to yield, for r^{4m-1}, m = 0 or 4/3 Beside the values, there are no other real zeros of . For m = 4/3 we have

$$S(r) = a_1 r^{4/3} + a_3 r^{-2/3} + a_5 r^{-8/3} + \ldots \tag{23}$$

where $a_1, a_3, \ldots a_{2n+1}$ (n = 0, 1, 2...) are the non-vanishing constants. From the result of the linear theory (Timoshenko and Goodier, 1970) in which the displacement R (r) is of the form.

$$R(r) = Ar + \frac{B}{r}, r \neq 0 \tag{24}$$

† Observe that the transformation (19) makes the present Eq. (20) look more hideous than Eq. (16). The argument in favour of (19) is that the linearization of the boundary conditions makes solution more tractable.

we let $S_0(r) \simeq S(r)$ and assume †

$$S_0(r) = a_1 r^{4/3} + a_3 r^{-2/3} \tag{25}$$

$$= r^{4/3}\left(a_1 + \frac{a_3}{r^2}\right)^{3/4} \tag{3.8}$$

With this, we have in the form

$$R_0(r) = r\left(a_1 + \frac{a_3}{r^2}\right)^{3/4} \tag{27}$$

The boundary conditions, applied to (27) yield

$$\left.\begin{aligned} a_1 &= \frac{1}{b^2 - a^2}\left[b^2 - a^2\left(\frac{\mu}{\mu + p}\right)^{1/3}\right] \\ a_3 &= \frac{2a^2b^2}{b^2 - a^2}\left[1 - \left(\frac{\mu}{\mu + p}\right)^{1/3}\right] \end{aligned}\right\} \tag{28}$$

On substituting (28) in (27) we obtain the radial displacement

$$R_0(r) = r\left[\frac{1}{b^2 - a^2}\left[\left(b^2 - a^2\left(\frac{\mu}{\mu + p}\right)^{1/3}\right) + 2a^2b^2\left(1 - \left(\frac{\mu}{\mu + p}\right)^{1/3}\frac{1}{r^2}\right)\right]\right]^{3/4} \tag{29}$$

In the section that follows we shall determine the components of stress.

COMPONENTS OF STRESS

We refer to (29). Let

$$\begin{aligned} \alpha &= \frac{1}{b^2 - a^2}\left[b^2 - a^2\left(\frac{\mu}{\mu + p}\right)^{1/3}\right] \\ \beta &= \frac{2a^2b^2}{b^2 - a^2}\left[1 - \left(\frac{\mu}{\mu + p}\right)^{1/3}\right] \end{aligned} \tag{30}$$

† The justification of the choice of the first two terms of (23) in (25) is on the basis of the rapid convergence of the series (23) to S_0 (r) for large r (>1), (see example (Chapra and Canale, 1988; Iheagwam and Onwuatu, 2000).

Thus

$$\left.\begin{aligned} R_0(r) &= r\left(\alpha+\frac{\beta}{r^2}\right)^{3/4} \\ \dot{R}_0(r) &= \left(\alpha+\frac{\beta}{r^2}\right)^{3/4}\left[1-\frac{3\beta}{2r^2}\left(\alpha+\frac{\beta}{r^2}\right)^{-1}\right] \\ R_0\dot{R}_0^{3}(r) &= r\left(\alpha+\frac{\beta}{r^2}\right)^{3}\left[\begin{aligned} &1-\frac{9\beta}{2r^2}\left(\alpha+\frac{\beta}{r^2}\right)^{-1}+\frac{27\beta^2}{4r^4}\left(\alpha+\frac{\beta}{r^2}\right)^{-1} \\ &-\frac{27\beta^3}{8r^6}\left(\alpha+\frac{\beta}{r^2}\right)^{-3} \end{aligned}\right] \\ R_0^3\dot{R}_0(r) &= r^3\left(\alpha+\frac{\beta}{r^2}\right)\left[1-\frac{3\beta}{2r^2}\left(\alpha+\frac{\beta}{r^2}\right)^{-1}\right] \end{aligned}\right\} \tag{31}$$

Substituting (31) in (14) accordingly we obtain the following components of stress

$$\sigma_{RR}(r)=\mu \left[1-\frac{8r^6}{8\left(\alpha r^2+\beta\right)^3-36\beta\left(\alpha r^2+\beta\right)^2+54\beta^2\left(\alpha r^2+\beta\right)-27\beta^3}\right] \tag{32}$$

$$\sigma_{\theta\theta}(r)=\mu\left[1-\frac{2r^6}{\left(\alpha r^2+\beta\right)^2\left[2\left(\alpha r^2+\beta\right)-3\beta\right]}\right] \tag{33}$$

CONCLUSION

We have shown that given an axi-symmetric plane strain deformation (4) of homogeneous isotropic compressible elastic (Blatz and Ko, 1962) material whose elastic potential is (9), the determination of the deformation gives and rise to a non-linear boundary-value problem (16, 18). We have also shown that a step to the solution to such a problem can be achieved by seeking a function which linearizes the boundary conditions, as achieved by (19) in the present case. This done, we relied on the method of series solution, being informed that function can posses a series representation, at least about its ordinary point (s). The radial displacement (29) obtained provides a working approximation. Finally, we were able to obtain the components of stress (32, 33) in a manner that engineers can apply with ease. It is hoped that this work will be extended to an anisotropic material.

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