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Hankel Determinant for Analytic Functions with Respect to Other Points

Gagandeep Singh
Department of Mathematics, DIPS College (Co-Educational),
Dhilwan, Kapurthala, Punjab, India

Abstract: This study is concerned with the estimate of second Hankel determinant for the classes of analytic-univalent functions with respect to conjugate points and with respect to symmetric conjugate points in the unit disc $E = \{z : |z| < 1\}$.

Key words: Analytic functions, starlike functions with respect to conjugate points, starlike functions, convex functions with respect to symmetric conjugate points, Hankel determinant, coefficient bounds

INTRODUCTION

Let, A be the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the unit disc $E = \{z : |z| < 1\}$. Let, S be the class of functions f (z)∈A and univalent in E. Sakaguchi (1959) introduced the class s_s^* consisting of functions of Eq. 1 and satisfying the condition:

$$\mathrm{Re}\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0, z \in E$$

The functions of the class s_s^* are called starlike functions with respect to symmetric points. Das and Singh (1977) defined the class K_S consisting of functions of Eq. 1 and satisfying the condition:

$$\mathrm{Re}\left\{\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right\} > 0, z \in E$$

The functions of the class K_S are known as convex functions with respect to symmetric points. Motivated from the research (Sakaguchi, 1959; Das and Singh, 1977) defined the following classes (El-Ashwah and Thomas, 1987):

$$S_c^* = \left\{f(z) \in A : \mathrm{Re}\left\{\frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}}\right\} > 0, z \in E\right\} \tag{2}$$

$$S_{sc}^* = \left\{f(z) \in A : \mathrm{Re}\left\{\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}}\right\} > 0, z \in E\right\} \tag{3}$$

Functions in the classes s_s^* are called starlike functions with respect to conjugate points and that in the class are known as starlike functions with respect to symmetric conjugate points.

Again Janteng *et al*. (2006a, b) introduced the following classes:

$$K_c = \left\{f(z) \in A : \mathrm{Re}\left\{\frac{2(zf'(z))'}{\left(f(z) + \overline{f(\bar{z})}\right)'}\right\} > 0, z \in E\right\} \tag{4}$$

$$K_{sc} = \left\{f(z) \in A : \mathrm{Re}\left\{\frac{2(zf'(z))'}{\left(f(z) - \overline{f(\bar{z})}\right)'}\right\} > 0, z \in E\right\} \tag{5}$$

Functions of the class K_c are convex functions with respect to conjugate points and that in the class K_c are called convex functions with respect to symmetric conjugate points.

Obviously, the functions in these classes are univalent. Various subclasses of analytic functions with respect to conjugate points and with respect to symmetric conjugate points were widely investigated by various researchers including Dahar and Janteng (2009), Selvaraj and Vasanthi (2011), Ravichandran (2004) and Tang and Deng (2013). Noonan and Thomas (1976) stated the qth Hankel determinant for q≥1 and n≥1 as:

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has also been considered by several researchers. For example, Noor (1983) determined the rate of growth of H_q (n) as $n\to\infty$ for functions given by Eq. 1 with bounded boundary. Ehrenborg (2000) studied the Hankel determinant of exponential polynomials. Also, Hankel determinant was studied by various researchers including Hayman (1958) and Pommerenke. Janteng *et al*. (2006a, b, 2007) studied the Hankel determinant for the classes of starlike functions, convex functions, starlike functions with respect to symmetric points, convex functions with respect to symmetric points. Recently, Singh (2013) established the hankel determinant for various subclasses of analytic functions with respect to symmetric points.

Easily, one can observe that the Fekete and Szego functional is H_2 (1). Fekete and Szego, then further generalised the estimate $|a^3-\mu a_2^2|$ where μ is real and f∈S. For the discussion in this study, researchers consider the Hankel determinant in the case of q = 2 and n = 2:

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$$

In this study, researchers established the sharp upper bound of the functional $|a_2\,a_4-a_3^2|$ for functions belonging classes s_c^*, s_{sc}^*, K_c and K_{sc}.

PRELIMINARY RESULTS

Let, P be the family of all functions p analytic in E for which Re (p(z))>0 and:

$$p(z)=1+p_1z+p_2z^2+\ldots \quad \text{for } z\in E \tag{6}$$

Lemma 1: If p∈ P, then $|p_k|\le 2$ (k = 1, 2, 3, ...).

Lemma 2: If p∈P, then:

$$2p_2=p_1^2+\left(4-p_1^2\right)x, 4p_3=p_1^3+2p_1\left(4-p_1^2\right)x- p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z$$

For some x and z satisfying $|x|\le 1$ and $p_1\in[0, 2]$ (Libera and Zlotkiewiez, 1982,1983).

RESULTS

Theorem 1: If f∈ s_c^*, then:

$$\left|a_2a_4-a_3^2\right|\le 1 \tag{7}$$

Proof: As, f∈ s_c^* so from Eq. 2:

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar z)}}=p(z) \tag{8}$$

On equating coefficients of z, z^2 and z^3 in the expansion of Eq. 8, researchers obtain:

$$\left\{\begin{matrix} a_2=p_1, a_3=\frac{p_2}{2}+\frac{p_1^2}{2} \\ a_4=\frac{p_3}{3}+\frac{p_1p_2}{2}+\frac{p_1^3}{6}\end{matrix}\right\} \tag{9}$$

Equation 9 yields:

$$a_2a_4-a_3^2\frac{1}{12}\left\{4p_1p_3-p_1^4-3p_2^2\right\} \tag{10}$$

Using Lemma 1 and 2 in Eq. 10, researchers obtain:

$$\left|a_2a_4-a_3^2\right|=\frac{1}{48}\left|\begin{matrix}-3p_1^4+2p_1^2\left(4-p_1^2\right)x-\left[12+p_1^2\right] \\ \left(4-p_1^2\right)x^2+8p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\end{matrix}\right|$$

Assume that p_1 = p and p∈[0, 2], using triangular inequality and $|z|\le 1$, researchers have:

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{48}\left\{\begin{matrix}3p^4+2p^2\left(4-p^2\right)|x|+\left[p^2+12\right] \\ \left(4-p^2\right)|x|^2+8p\left(4-p^2\right)\left(1-|x|^2\right)\end{matrix}\right\}$$

or;

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{48}\left\{\begin{matrix}3p^4+8p\left(4-p^2\right)+2p^2\left(4-p^2\right) \\ \delta+\left[p^2-8p+12\right]\left(4-p^2\right)\delta^2\end{matrix}\right\}=\frac{1}{48}F(\delta)$$

Where:

$$\delta=|x|\le 1$$

And:

$$F(\delta)=3p^4+8p\left(4-p^2\right)+2p^2\left(4-p^2\right) \delta+\left[p^2-8p+12\right]\left(4-p^2\right)\delta^2$$

is an increasing function. Therefore:

$$\operatorname{Max}F(\delta)=F(1)=48$$

Hence:

$$\left|a_2a_4-a_3^2\right|\le 1$$

The result is sharp for:

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}=\frac{1+z}{1-z}$$

And:

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}=\frac{1+z^2}{1-z^2}$$

On the same lines, researchers can easily prove the following theorem.

Theorem 2: If f∈K_c, then:

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{8}$$

The result is sharp for $p_1 = 1$, $p_2 = -1$ and $p_3 = -2$.

Theorem 3: If f∈ s_{sc}^*, then:

$$\left|a_2a_4-a_3^2\right|\le 1 \tag{11}$$

Proof: As f∈ s_{sc}^*, so from Eq. 3:

$$\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}=p(z) \tag{12}$$

On equating coefficients of z, z^2 and z^3 in the expansion of Eq. 12, researchers obtain:

$$\left\{\begin{matrix} a_2=\frac{P_1}{2}, a_3=\frac{P_2}{2} \\ a_4=\frac{P_3}{4}+\frac{P_1P_2}{8} \end{matrix}\right\} \tag{13}$$

Equation 13 yields:

$$a_2a_4-a_3^2=\frac{1}{16}\left\{2p_1p_3+p_1^2p_2-4p_2^2\right\} \tag{14}$$

Using Lemma 1 and 2 in Eq. 14, researchers obtain:

$$\left|a_2a_4-a_3^2\right|=\frac{1}{32}\left|\begin{matrix}-p_1^2\left(4-p_1^2\right)x-\left[8-p_1^2\right]\left(4-p_1^2\right) \\ x^2+2p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\end{matrix}\right|$$

Assume that p_1= p and p∈[0, 2], using triangular inequality and $|z|\le 1$, researchers have:

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{32}\left\{\begin{matrix}2p\left(4-p^2\right)+p^2\left(4-p^2\right)\delta+ \\ \left[-p^2-2p+8\right]\left(4-p^2\right)\delta^2\end{matrix}\right\}=\frac{1}{32}(F\delta)$$

Where:

$$\delta=|x|\le 1$$

And:

$$F(\delta)=2p\left(4-p^2\right)+p^2\left(4-p^2\right)\delta+ \left[-p^2-2p+8\right]\left(4-p^2\right)\delta^2$$

is an increasing function. Therefore:

$$\text{Max}F(\delta)=F(1)$$

Consequently:

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{32}G(p) \tag{15}$$

Where:

$$G(p)=F(1)$$

So:

$$G(p)=-8p^2+32$$

And:

$$\text{Max } G(p)=G(0)$$

Hence from Eq. 15, researchers obtain Eq. 11. The result is sharp for $p_1 = 0$, $p_2 = 2$ and $p_3 = 0$. Using the above technique, the proof of the following theorem is obvious.

Theorem 4: If f∈K_{sc}, then:

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{9}$$

The result is sharp for $p_1 = 0$, $p_2 = 2$ and $p_3 = 0$.

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