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On Size Biased Exponential Distribution

[1]K.A. Mir, [2]A. Ahmed and [2]J.A. Reshi
[1]Department of Statistics, Government Degree College Bemina Srinagar, Kashmir, India
[2]Department of Statistics, University of Kashmir, India

Abstract: In this study, a Size Biased Exponential Distribution (SBEPD) is introduced and its moments are obtained. A size biased exponential distribution; a particular case of the weighted exponential distribution, taking the weights as the variate values has been defined. The estimates of the parameters of Size Biased Exponential Distribution (SBEPD) are obtained by employing the method of moments, maximum likelihood estimator and Bayesian estimation. Also, a Bayes' estimator of Size Biased Exponential Distribution (SBEPD) has been obtained by using non-informative and gamma prior distributions.

Key words: Exponential distribution, size biased exponential distribution, moment estimator, maximum likelihood estimator, Bayes' estimator

INTRODUCTION

Size biased distributions are a special case of the more general form known as weighted distributions. First introduced by Fisher (1934) to model ascertainment bias, these were later formalized in a unifying theory (Patil and Rao, 1977). These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non-experimental, non-replicated and non-random categories. Lloyd (1952) described a method for obtaining the Best Linear Unbiased Estimators (BLUE's) of the parameters of a distribution , using order ststistics.

Epstein and Sobel (1953) published a paper which presented the maximum likelihood estimator of the scale parameter (σ), of the one parameter exponential distribution in the case of censoring from the right. Epstein (1960) extended his owns results to estimator of σ and θ for the one parameter exponential distribution in the case of censoring from the right and/or left. A brief list of researchers and their substantial researches can be seen in Johnson *et al*. (1995) and Johnson and Kotz (1969). Gove (2003) reviewed some of the more recent results on size biased distributions pertaining to parameter estimation in forestry. Mir (2007) and Mir and Ahmad (2009) also discussed some of the size biased distributions.

If the random variable X has distribution f (x; θ), with unknown parameter θ then the corresponding size biased distribution is of the form:

$$f^*(x;\theta) = \frac{x^c f(x;\theta)}{\mu'_c} \tag{1}$$

For continuous series:

$$\mu'_c = \int x^c f(x;\theta)dx \tag{2i}$$

For discrete series:

$$\mu'_c = \sum_{i=1}^{n} x^c f(x;\theta)dx \tag{2ii}$$

A continuous random variable X is said to be exponential distribution with parameter θ and its probability density function is given as:

$$\begin{aligned}&\text{If}\quad X \sim \exp(\theta)\\ &\text{Then,}\quad f(x;\theta) = \theta e^{-\theta x}\\ &\qquad\qquad = 0,\quad \text{otherwise } \theta > 0\end{aligned} \tag{3}$$

Therefore:

$$\mu'_1 = \int_0^\infty x f(x;\theta)dx = \frac{1}{\theta}$$

$$\mu'_2 = \int_0^\infty x^2 f(x;\theta)dx = \frac{2}{\theta^2}$$

$$\mu_2 = \frac{1}{\theta^2}$$

Case 2:

$$\begin{aligned}&\text{If } X \sim \exp(\theta)\\ &\text{Then,}\quad f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}\\ &\qquad\qquad = 0,\quad \text{otherwise}\ \theta > 0\end{aligned} \tag{4}$$

Corresponding Author: K.A. Mir, Department of Statistics, Government Degree College Bemina Srinagar, Kashmir, India

Therefore:

$$\mu_1' = \int_0^\infty x f(x;\theta)dx = \theta$$

$$\mu_2' = \int_0^\infty x^2 f(x;\theta)dx = 2\theta^2$$

$$\mu_2 = \theta^2$$

In this study, a Size Biased Exponential Distribution (SBEPD) is defined. Moments about origin is obtained. The estimates of the parameters of Size Biased Exponential Distribution (SBEPD) are obtained by employing the moments, maximum likelihood and Bayesian Method of estimation.

SIZE BIASED EXPONENTAIL DISTRIBUTION

Case 1: A Size Biased Exponential Distribution (SBEPD) is obtained by applying the weights x^c where c = 1 to the exponential distribution. Researchers have from Eq. 1 and 3:

$$\mu_1' = \int_0^\infty x f(x;\theta)dx = \frac{1}{\theta}$$

$$\mu_1' = \int_0^\infty x\theta e^{-\theta x}dx = \frac{1}{\theta}$$

$$\int_0^\infty x\theta^2 e^{-\theta x}\,dx = 1$$

$$\Rightarrow \int_0^\infty f(x;\theta) = 1$$

Where, f (x;θ) represents a probability density function. This gives the Size Biased Exponential Distribution (SBEPD) as:

$$f(x;\theta) = \theta^2 x e^{-\theta x} \tag{5}$$

Case 2: A Size Biased Exponential Distribution (SBEPD) is obtained by applying the weights x^c, where c = 1 to the exponential distribution. Researchers have from Eq. 1 and 4:

$$\mu_1' = \int_0^\infty x f(x;\theta)dx = \theta$$

$$\mu_1' = \int_0^\infty x\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx = \theta$$

$$\mu_1' = \int_0^\infty x\frac{1}{\theta^2}e^{-\frac{x}{\theta}}\,dx = 1$$

$$\int_0^\infty x\theta^{-2}\, e^{-\frac{x}{\theta}}dx = 1$$

$$\Rightarrow \int_0^\infty f(x;\theta) = 1$$

Where, f(x; θ) represents a probability density function. This gives the Size Biased Exponential Distribution (SBEPD) as:

$$f(x;\theta) = \theta^{-2}\, x e^{-\frac{x}{\theta}} \tag{6}$$

Mean and variance by the method of moments: The rth moment of size biased exponential distribution (Eq. 13) about origin is obtained as:

$$\mu_r' = E(x)^r = \int_0^\infty x^r f(x;\theta)dx$$

$$= \int_0^\infty x^r\theta^2 x e^{-\theta x}dx = \int_0^\infty x^{r+1}\theta^2 e^{-\theta x}dx \tag{7}$$

$$\mu_r' = \int_0^\infty x^{(r+2)-1}\theta^2 e^{-\theta x}dx = \theta^2\frac{\Gamma r+2}{\theta^{r+2}}$$

Put r = 1, 2 in Eq. 7, researchers get:

$$\mu_1' = \theta^2\frac{\Gamma 3}{\theta^3} = \frac{2}{\theta} \tag{8}$$

Equation 8 is the mean of size biased exponential distribution:

$$\mu_2' = \theta^2\frac{\Gamma 4}{\theta^4} = \frac{6}{\theta^2}$$

$$\mu_2 = \frac{2}{\theta^2} \tag{9}$$

Equation 9 is the variance of size biased exponential distribution.

Mean and variance by the method of moments: The rth moment of size biased exponential distribution (Eq. 13) about origin is obtained as:

$$\mu_r' = E(x)^r = \int_0^\infty x^r f(x;\theta)dx = \int_0^\infty x^r\frac{1}{\theta^2}x e^{-\frac{x}{\theta}}dx$$

$$= \int_0^\infty \frac{1}{\theta^2}x^{r+1}e^{-\frac{-x}{\theta}}dx = \int_0^\infty \frac{1}{\theta^2}x^{(r+2)-1}e^{-\frac{-x}{\theta}}dx \tag{10}$$

$$= \theta^{-2}\frac{\Gamma r+2}{\left(\frac{1}{\theta}\right)^{r+2}}$$

Put r = 1, 2 in Eq. 10, researchers get:

$$\mu_1' = \theta^{-2}\frac{\Gamma 3}{\left(\frac{1}{\theta}\right)^3} = 2\theta \tag{11}$$

Equation 11 is the mean of the size biased exponential distribution:

$$\mu_2' = \theta^{-2}\frac{\Gamma 4}{\left(\frac{1}{\theta}\right)^4} = 6\theta^2 = 2\theta^2 \quad (12)$$

Equation 12 is the variance of the size biased exponential distribution.

ESTIMATION OF PARAMETERS

In this study, researchers discuss the various estimation methods for size biased gamma distribution and verifying their efficiencies.

Methods of moments: In the method of moments replacing the population mean and variance by the corresponding sample mean and variance, researchers have:

Case 1:

$$\mu_1' = \bar{x};\ \frac{2}{\theta} = \bar{x}$$

$$\text{Then}\quad \hat{\theta} = \frac{2}{\bar{x}} \quad (13)$$

$$\text{And, if}\ \mu_2 = S^2;\ \frac{2}{\theta^2} = S^2$$

Then:

$$\hat{\theta} = \frac{\sqrt{2}}{S} \quad (14)$$

Equation 13 and 14 are the estimates of size biased exponential distribution.

Case 2:

$$\mu_1' = \bar{x}$$

$$\text{Then}\quad 2\theta = \bar{x};\ \hat{\theta} = \frac{\bar{x}}{2} \quad (15)$$

$$\text{And, if}\ \mu_2 = S^2;\ 2\theta^2 = S^2$$

Then:

$$\hat{\theta} = \frac{S}{\sqrt{2}} \quad (16)$$

The estimates in Eq. 15 and 16 are also the estimates of size biased exponential distribution.

Method of maximum likelihood estimator: Let x_1, x_2, x_3,...,x_n be a random sample from the size biased exponential distribution then the corresponding likelihood function is given as:

Case 1:

$$X \sim \exp(\theta)$$

$$f(x;\theta) = \theta^2 x e^{-\theta x} \quad (17)$$

$$\Rightarrow\quad L(x;\theta) = \theta^{2n}\prod_{i=1}^{n} x_i e^{-\theta\sum_{i=1}^{n} x_i}$$

The log likelihood of Eq. can be written as:

$$\text{LogL} = 2n\log\theta + \log\sum_{i=1}^{n} xi - \theta\sum_{i=1}^{n} xi$$

The corresponding likelihood equation is given as:

$$\frac{2n}{\theta} - \sum_{i=1}^{n} xi = 0,\ \frac{2n}{\theta} = \sum_{i=1}^{n} xi \quad (18)$$

$$\Rightarrow\quad \hat{\theta} = \frac{2}{\bar{x}}$$

This coincides with moment estimate.

Case 2:

$$\text{If}\qquad X \sim \exp(\theta)$$

$$\text{Then,}\quad f(x;\theta) = \theta^{-2} x e^{\frac{-x}{\theta}} \quad (19)$$

$$L(x;\theta) = \theta^{-2n}\prod_{i=1}^{n} xi\, e^{\frac{-\sum_{i=1}^{n} x_i}{\theta}}$$

The log likelihood of Eq. 19 can be written as:

$$\text{LogL} = -2n\log\theta + \log\sum_{i=1}^{n} x_i - \frac{\sum_{i=1}^{n} x_i}{\theta}$$

The corresponding likelihood equation is given as:

$$-\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} xi}{\theta^2} = 0$$

$$\frac{2n\theta}{\theta^2} = \frac{\sum_{i=1}^{n} xi}{\theta^2} \quad (20)$$

$$\Rightarrow\quad \hat{\theta} = \frac{\bar{x}}{2}$$

This coincides with moment estimate.

Bayesian estimation of parameter of Size Biased Exponential Distribution (SBEPD)

Case 1: The likelihood function of SBEPD Eq. 5 is given as:

$$L(x;\theta)=\theta^{2n}\prod_{i=1}^{n}x_i\, e^{-\theta\sum_{i=1}^{n}x_i}$$

$$L(x;\theta)=\theta^{2n}\prod_{i=1}^{n}x_i\, e^{-\theta y}$$

Where:

$$y=\sum_{i=1}^{n}xi$$

Since 0<θ<∞, therefore researchers assume that prior information about θ. Researchers know that g(θ) proportional to 1/θ. Thus:

$$g(\theta)=\frac{1}{\theta};\ \theta>0$$

The posterior distribution is given as:

$$\Pi(\theta/y)=\frac{Lg(\theta)}{\int_0^\infty Lg(\theta)d\theta}$$

$$=\frac{\frac{1}{\theta}\theta^{2n}\prod_{i=1}^{n}x_i\, e^{-\theta y}}{\int_0^\infty \frac{1}{\theta}\theta^{2n}\prod_{i=1}^{n}x_i\, e^{-\theta y}d\theta}$$

Bayes' estimator of θ:

$$\hat{\theta}=\int_0^\infty \theta\Pi(\theta/y)d\theta$$

$$=\int_0^\infty \theta\frac{\frac{1}{\theta}\theta^{2n}\prod_{i=1}^{n}x_i e^{-\theta y}}{\int_0^\infty \frac{1}{\theta}\theta^{2n}\prod_{i=1}^{n}x_i e^{-\theta y}d\theta}d\theta \tag{21}$$

$$=\int_0^\infty \theta\frac{\theta^{2n}e^{-\theta y}}{\int_0^\infty \theta^{2n-1}e^{-\theta y}d\theta}d\theta=\frac{2n}{y}=\frac{2}{x}$$

This coincides with moment estimator and maximum likelihood estimator.

Case 2: The likelihood function of SBEPD Eq. 6 is given as:

$$L(x;\theta)=\theta^{-2n}\prod_{i=1}^{n}x_i e^{\frac{-\sum_{i=1}^{n}x_i}{\theta}}$$

$$=\theta^{-2n}\prod_{i=1}^{n}x_i\, e^{-\frac{y}{\theta}}$$

Where;

$$y=\sum_{i=1}^{n}x_i$$

Since 0<θ<∞, therefore researchers assume that prior information about θ. Researchers know that g (θ) proportional to 1/θ. Thus:

$$g(\theta)=\frac{1}{\theta};\ \theta>0$$

The posterior distribution is given as:

$$\Pi(\theta/y)=\frac{Lg(\theta)}{\int_0^\infty Lg(\theta)d\theta}$$

$$=\frac{\frac{1}{\theta}\theta^{-2n}\prod_{i=1}^{n}x_i\, e^{\frac{-y}{\theta}}}{\int_0^\infty \frac{1}{\theta}\theta^{-2n}\prod_{i=1}^{n}x_i\, e^{\frac{-y}{\theta}}d\theta}$$

Bayes' estimator of θ:

$$\hat{\theta}=\int_0^\infty \theta\Pi(\theta/y)d\theta$$

$$\hat{\theta}=\int_0^\infty \theta\frac{\frac{1}{\theta}\theta^{2n}\prod_{i=1}^{n}x_i e^{\frac{-y}{\theta}}}{\int_0^\infty \frac{1}{\theta}\theta^{2n}\prod_{i=1}^{n}x_i e^{\frac{-y}{\theta}}d\theta}d\theta$$

$$\hat{\theta}=\int_0^\infty \theta\frac{\theta^{2n}e^{\frac{-y}{\theta}}}{\int_0^\infty \theta^{2n-1}e^{\frac{-y}{\theta}}d\theta}d\theta \tag{22}$$

$$\hat{\theta}=\frac{y}{2n}=\frac{\overline{x}}{2}$$

Equation 22 coincides with moment and maximum likelihood estimator.

CONCLUSION

The estimates of the parameters of Size Biased Exponential Distribution (SBEPD) are obtained by employing different estimation methods. A Bayes' estimator of Size Biased Exponential Distribution (SBEPD) has also been obtained by using non-informative and gamma prior distributions.

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