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Oscillations in Non-Autonomous Neutral Impulsive Differential Equations with Several Delays

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Abstract: In this study, we establish some sufficient conditions for the oscillation of all the solutions of the non-autonomous neutral impulsive differential equation with several delays

$$\begin{cases} \left[x(t)-\sum_{j=1}^{M} p_j(t)x(t-\tau_j(t))\right]' + \sum_{i=1}^{N} q_i(t)x(t-\sigma_i(t)) = 0,\ t \neq t_k \\ \Delta\left[x(t_k)-\sum_{j=1}^{M} p_j(t_k)x(t_k-\tau_j(t_k))\right] + \sum_{i=1}^{N} q_{ik}x(t_k-\sigma_i(t_k)) = 0, \end{cases}$$

where, t, $t_k \geq t_0$. The study was carried out under the assumption that for sufficiently large t, the coefficients of the equation satisfy the conditions

$$\sum_{j=1}^{M}\left(p_j(t)+p_j(t_k)\right) \leq 1 \text{ and } \sup_{1\leq k<\infty}\left\{\sum_{i=1}^{N}\left(q_i(s)+q_{ik}\right) : s>t\right\} > 0$$

Key words: Non-autonomous neutral impulsive equations, sufficient oscillatory conditions, dependence of conditions on coefficients and delays, characteristic equation, characteristic roots

INTRODUCTION

The problem of oscillatory and asymptotic behaviours of solutions of neutral impulsive differential equations of the first order is of both theoretical and practical importance (Bainov and Simeonov, 1995; Berezansky and Braverman, 2003; Candan and Dahiya, 2005; Jankowski, 2007). One of the major reasons may be due to the fact that equations of this type abound in networks containing lossless transmission lines. Such networks are found in high speed computers where, lossless transmission lines are used to interconnect switching circuits. We must also acknowledge the role of such equations in the motion of radiating electrons, population growth, the spread of epidemics, to mention just a few (Gyori and Ladas, 1991).

The aim of this study is to obtain sufficient conditions which depend only on the coefficients and delays for the oscillation of all solutions of equation of the form:

$$\begin{cases} \left[x(t)-\sum_{j=1}^{M} p_j(t)x(t-\tau_j(t))\right]' \\ +\sum_{i=1}^{N} q_i(t)x(t-\sigma_i(t)) = 0,\ t \neq t_k \\ \Delta\left[x(t_k)-\sum_{j=1}^{M} p_j(t_k)x(t_k-\tau_j(t_k))\right] \\ +\sum_{i=1}^{N} q_{ik}x(t_k-\sigma_i(t_k)) = 0, \end{cases} \quad (1.1)$$

where, t, $t_k \geq t_0$.

The advantage of working with these conditions rather than with the characteristic equation of the neutral impulsive equation under consideration is that they are explicit and are therefore easily verifiable, while the determination of whether or not a real root to the characteristic equation exists may be quite a problem itself.

As is customary, a solution x of an impulsive differential equation is said to be:

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- Finally positive, if there exists $T \geq 0$ such that x(t) is defined for $t \geq T$ and $x(t)>0$ for all $t \geq T$
- Finally negative, if there exists $T \geq 0$ such that x(t) is defined for $t \geq T$ and x(t) for all $t \geq T$
- Non-oscillatory, if it is either finally positive or finally negative
- Oscillatory, if it is neither finally positive nor finally negative
- Regular, if it is defined in some half line (T_x, ∞) for some $T_x \in R$ and $\sup\{|x(t)|: t \geq T\} > 0$, $\forall T > T_x$ (Lakshmikantham *et al.*, 1989)

Usually, the solution x(t) for $t \in (t_0, T)$ of the impulsive differential equation or its first derivative x'(t) is a piece-wise continuous function with points of discontinuity $t_k \in (t_0, T)$, $t_k \neq t$. Therefore, in order to simplify the statements of our assertions later, we introduce the set of functions PC and PC^r, which are defined as follows:

Let $r \in N$, $D := (T, \infty) \subset R$ and let $S := \{t_k\}_{k\in E}$, where, E represents a subscript set, which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout the discussion, we will assume that the sequence $\{t_k\}_{k\in E}$ are moments of impulse effect and satisfy the properties:

C1.1: If $\{t_k\}_{k\in E}$ is defined with E: = N, then $0<t_1<t_2<\cdots$ and $\lim_{k\to+\infty} t_k = +\infty$.

C1.2: If $\{t_k\}_{k\in E}$ is defined with E: = Z, then $t_0 \leq 0 < t_1$, $t_k < t_{k+1}$ for all $k \in Z$, $k \neq 0$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$.

We denote by PC (D, R) the set of all functions φ: D→R, which is continuous for all $t \in D$, $t \notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$.

By Pc^r (D, R), we denote the set of functions φ: D→R having derivative $d^j\varphi/dt^j \in PC(D, R)$, $0 \leq j \leq r$ (Bainov and Simeonov, 1998; Lakshmikantham *et al.*, 1989).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols PC(D, R; S) and PC^r(D, R; S), $r \in N$.

In the sequel, all functional inequalities that we write are assumed to hold finally that is, for all sufficiently large t.

Statement of the problem: The following results, which are essential in the proofs of the theorems have been extracted from studies by Bainov and Simeonov (1998) and Gyori and Ladas (1991).

Lemma 2.1: Let $a \in (-\infty, 0)$, $\tau \in (0, \infty)$, $t_0 \in R$ and suppose the function $x \in C((t_0-\tau, \infty), R)$ satisfies the inequality:

$$x(t) \leq a + \max_{t-\tau\leq s\leq t} x(s) \text{ for } t \geq t_0 \tag{2.1}$$

Then, x cannot be a non-negative function. Consider the impulsive differential equation:

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \ t \notin S \\ \Delta x(t_k) + \sum_{i=1}^{n} p_{i0} x(t_k - \tau_i) = 0, \ \forall t_k \in S, \end{cases} \tag{2.2}$$

where, $\tau_i \geq 0$; p_i, $p_{i0} \in R$, $1 \leq i \leq n$.

Theorem 2.1: Assume that τ_i p_i, $p_{i0} \geq 0$ for $1 \leq i \leq n$. Then, the condition

$$\sum_{i=1}^{n} (p_i + p_{i0})\tau_i > e^{-1}$$

is sufficient for the oscillation of all the solutions of Eq. 2.2.

Now consider the impulsive delay differential equation

$$\begin{cases} x'(t) + \sum_{i=1}^{n} q_i(t) x(t-\tau_i(t)) = 0, \ t \notin S \\ \Delta x(t_k) + \sum_{i=1}^{n} q_{ik} x(t_k - \tau_i(t_k)) = 0, \ \forall t_k \in S \end{cases} \tag{2.3}$$

and the impulsive delay inequalities

$$\begin{cases} y'(t) + \sum_{i=1}^{n} p_i(t) y(t-\tau_i(t)) \leq 0, \ t \notin S \\ \Delta y(t_k) + \sum_{i=1}^{n} p_{ik} y(t_k - \tau_i(t_k)) \leq 0, \ \forall t_k \in S \end{cases} \tag{2.4}$$

and

$$\begin{cases} z'(t) + \sum_{i=1}^{n} r_i(t) z(t-\tau_i(t)) \geq 0, \ t \notin S \\ \Delta z(t_k) + \sum_{i=1}^{n} r_{ik} z(t_k - \tau_i(t_k)) \geq 0, \ \forall t_k \in S. \end{cases} \tag{2.5}$$

We introduce the condition:

$$\text{C2.1} \begin{cases} p_i, q_i, r_i \in PC(R_+, R_+), \ \tau_j \in C(R_+, R_+), \ i = 1, 2, \cdots, n \\ p_{ik}, q_{ik}, r_{ik} \geq 0, \ k \in N, \ i = 1, 2, \cdots, n. \end{cases}$$

Let, $t_0 \geq 0$. The initial interval associated with the above equation and inequalities is interval (t_{-1}, t_0), where,

$$t_{-1} = \min_{1\le i\le n}\left\{\inf_{t\ge t_0}\{t-\tau_i(t)\}\right\} \tag{2.6}$$

Theorem 2.2: Let condition C2.1 be fulfilled. Suppose further that:

$$\text{C2.2}\ \begin{cases} p_i(t)\ge q_i(t)\ge r_i(t);\ \forall t\in R_+,\ i=1,\ 2,\cdots,n \\ p_{ik}\ge q_{ik}\ge r_{ik};\ k\in N,\ i=1,\ 2,\cdots,n. \end{cases}$$

Assume that x, y and z are solutions of Eq. 2.3 and inequalities 2.4 and 2.5, respectively and belong to the space PC($[t_{-1}, +\infty)$, R) and such that:

$$y(t)>0,\ t\ge t_0 \tag{2.7}$$

$$z(t_0^+)\ge x(t_0^+)\ge y(t_0^+) \tag{2.8}$$

$$\frac{y(t)}{y(t_0)}\ge\frac{x(t)}{x(t_0)}\ge\frac{z(t)}{z(t_0)}\ge 0,\ t_{-1}\le t\le t_0 \tag{2.9}$$

Then,

$$z(t)\ge x(t)\ge y(t),\ \forall t\ge t_0 \tag{2.10}$$

Consider now the impulsive differential inequality 2.4 together with the impulsive differential equation:

$$\begin{cases} x'(t)+\sum_{i=1}^{n}p_i(t)x(t-\tau_i(t))=0,\ t\notin S \\ \Delta x(t_k)+\sum_{i=1}^{n}p_{ik}x(t_k-\tau_i(t_k))=0,\ \forall t_k\in S \end{cases} \tag{2.11}$$

From Theorem 2.2, we obtain the following:

Corollary 2.1: Let condition C2.1 be fulfilled. Then, the following statements are equivalent:

- Inequality 2.4 has a finally positive solution
- Equation 2.11 has a finally positive solution

Lemma 2.2: Let us now be given a non-autonomous neutral impulsive differential equation with several delays

$$\begin{cases} \left[x(t)-\sum_{j=1}^{M}p_j(t)x(t-\tau_j(t))\right]' + \\ \qquad +\sum_{i=1}^{N}q_i(t)x(t-\sigma_i(t))=0,\ t\notin S \\ \Delta\left[x(t_k)-\sum_{j=1}^{M}p_j(t_k)x(t_k-\tau_j(t_k))\right]+ \\ \qquad +\sum_{i=1}^{N}q_{ik}x(t_k-\sigma_i(t_k))=0,\ \forall t_k\in S \end{cases} \tag{2.12}$$

where, $t\in(t_0, T)/S$, $t_k\in S$ for $1\le k<\infty$. We introduce the following conditions:

$$\text{C2.2}\ \begin{cases} p_j\in PC^1([t_0,\infty),R_+),\ \tau_j\in C^1 \\ \quad ([t_0,\infty),R_+);\ 1\le j\le M, \\ q_i\in PC([t_0,\infty),R_+),\ \sigma_i\in C \\ \quad ([t_0,\infty),R_+);\ 1\le i\le N \end{cases}$$

for all $1\le j\le M$ and $1\le i\le N$,

$$\text{C2.3}\quad \sup_{t\ge t_0}\tau_j(t)<+\infty,\ \sup_{t_k\ge t_0}\tau_j(t_k)<+\infty$$

and

$$\text{C2.4}\quad \sup_{t\ge t_0}\sigma_i(t)<+\infty,\ \sup_{t_k\ge t_0}\sigma_i(t_k)<+\infty$$

Assume that conditions C2.2-C2.4 and relations

$$\begin{gathered} \sum_{j=1}^{M}\left(p_j(t)+p_j(t_k)\right)\le 1 \text{ and} \\ \sup_{1\le k<\infty}\left\{\sum_{i=1}^{N}\left(q_i(s)+q_{ik}\right):s>t\right\}>0 \end{gathered} \tag{2.13}$$

are satisfied for all $t\in(t_0, T)/S$ and $t_k\in S$, $\forall k\in N$. Let x be a finally positive solution of Eq. 2.12 and set

$$v(t)=x(t)-\sum_{j=1}^{M}p_j(t)x(t-\tau_j(t)) \tag{2.14}$$

Then, v is finally positive, non-increasing and

$$\begin{cases} v'(t)=-\sum_{i=1}^{N}q_i(t)x(t-\sigma_i(t)),\ t\notin S \\ \Delta v(t_k)=-\sum_{i=1}^{N}q_{ik}x(t_k-\sigma_i(t_k)),\ \forall t_k\in S. \end{cases} \tag{2.15}$$

RESULTS AND DISCUSSION

In this study, we show that if conditions C2.2-C2.4 and inequalities (2.13) are satisfied, then every solution of Eq. 2.12 oscillates provided the same is true for the non-neutral impulsive equation:

$$\begin{cases} y'(t)+\sum_{i=1}^{N}q_i(t)y(t-\sigma_i(t))=0,\ t\notin S \\ \Delta y(t_k)+\sum_{i=1}^{N}q_{ik}y(t_k-\sigma_i(t_k))=0,\ \forall t_k\in S \end{cases} \tag{3.1}$$

Theorem 3.1: Assume that conditions C2.2-C2.4 and inequalities (2.13) are fulfilled and suppose that every solution of Eq. 3.1 oscillates, then every solution of Eq. 2.12 also oscillates.

Proof: Assume conversely that Eq. 2.12 has a finally positive solution x. Then, by Lemma 2.2, v is finally positive. Also, $x(t)>v(t)$ for $t\in(t_0, T)/S$ and so (Eq. 2.15) yields

$$\begin{cases} v'(t)+\sum_{i=1}^{N} q_i(t)v(t-\sigma_i(t))\le 0,\ t\notin S \\ \Delta v(t_k)+\sum_{i=1}^{N} q_{ik}v(t_k-\sigma_i(t_k))\le 0,\ \forall t_k\in S \end{cases} \tag{3.2}$$

By corollary 2.1, it follows that Eq. 3.1 also, has a finally positive solution, which leads to a contradiction. This completes the proof of Theorem 3.1.

Remark 3.1: Notice that it is possible to use Theorem 3.1 together with any explicit sufficient conditions for the oscillation of all solutions of Eq. 3.1 to obtain explicit sufficient conditions for the oscillation of all solutions of Eq. 2.12.

We now describe a technique, which can be used to obtain successively improved oscillation results for Eq. 2.12. In the anticipated procedure, Theorem 3.1 may be thought of as being the first theorem. The second theorem in this succession is obtained in what follows.

Let us substitute Eq. 2.14 into 2.15 to obtain

$$\begin{cases} v'(t)+\sum_{i=1}^{N} q_i(t)v(t-\sigma_i(t))+\sum_{i=1}^{N} q_i(t)\sum_{j=1}^{M} p_j(t-\sigma_i(t))* \\ \quad *x(t-\sigma_i(t)-\tau_j(t-\sigma_i(t)))=0,\ t\notin S, \\ \Delta v(t_k)+\sum_{i=1}^{N} q_{ik}v(t_k-\sigma_i(t_k))+\sum_{i=1}^{N} q_{ik}\sum_{j=1}^{M} p_j(t_k-\sigma_i(t_k))* \\ \quad *x(t_k-\sigma_i(t_k)-\tau_j(t_k-\sigma_i(t_k)))=0,\ \forall t_k\in S \end{cases} \tag{3.3}$$

Under the hypotheses of Lemma 2.2, we have

$$0<v(t)<x(t),\ t\in[t_0,T)\setminus S$$

and so (Eq. 3.3) yields the inequality

$$\begin{cases} v'(t)+\sum_{i=1}^{N} q_i(t)v(t-\sigma_i(t))+\sum_{i=1}^{N} q_i(t)\sum_{j=1}^{M} p_j(t-\sigma_i(t))* \\ \quad *v(t-\sigma_i(t)-\tau_j(t-\sigma_i(t)))\le 0,\ t\notin S, \\ \Delta v(t_k)+\sum_{i=1}^{N} q_{ik}v(t_k-\sigma_i(t_k))+\sum_{i=1}^{N} q_{ik}\sum_{j=1}^{M} p_j(t_k-\sigma_i(t_k))* \\ \quad *v(t_k-\sigma_i(t_k)-\tau_j(t_k-\sigma_i(t_k)))\le 0,\ \forall t_k\in S \end{cases}$$

The following result, which improves Theorem 3.1, is now obvious.

Theorem 3.2: Assume that conditions C2.2-C2.4 and inequalities (2.13) are satisfied and suppose that every solution of the equation

$$\begin{cases} y'(t)+\sum_{i=1}^{N} q_i(t)y(t-\sigma_i(t))+\sum_{i=1}^{N} q_i(t)\sum_{j=1}^{M} p_j(t-\sigma_i(t))* \\ \quad *y(t-\sigma_i(t)-\tau_j(t-\sigma_i(t)))=0,\ t\notin S, \\ \Delta y(t_k)+\sum_{i=1}^{N} q_{ik}y(t_k-\sigma_i(t_k))+\sum_{i=1}^{N} q_{ik}\sum_{j=1}^{M} p_j(t_k-\sigma_i(t_k))* \\ \quad *y(t_k-\sigma_i(t_k)-\tau_j(t_k-\sigma_i(t_k)))=0,\ \forall t_k\in S \end{cases}$$

oscillates, then every solution of Eq. 2.12 also oscillates.

The following result is an immediate consequence of Theorems 2.1 and 3.2 applied to the neutral impulsive equation with constant coefficients and constant delays:

$$\begin{cases} \left[x(t)-\sum_{j=1}^{M} p_j x(t-\tau_j)\right]'+\sum_{i=1}^{N} q_i x(t-\sigma_i)=0,\ t\notin S \\ \Delta\left[x(t_k)-\sum_{j=1}^{M} p_j x(t_k-\tau_j)\right]+\sum_{i=1}^{N} q_{i0}x(t_k-\sigma_i)=0,\ \forall t_k\in S \end{cases} \tag{3.4}$$

Corollary 3.1: Assume that the coefficients and the delays of Eq. 3.4 are non-negative real numbers such that

$$\sum_{j=1}^{M}(p_j+p_{j0})\le 1$$

and

$$\left(\sum_{i=1}^{N}(q_i+q_{i0})\sigma_i\right)\left(1+\sum_{j=1}^{M}(p_j+p_{j0})\right)+$$
$$+\left(\sum_{i=1}^{N}(q_i+q_{i0})\right)\left(\sum_{j=1}^{M}(p_j+p_{j0})\tau_j\right)>e^{-1},\ t\in[t_0,T)\setminus S$$

Then, every solution of Eq. 3.4 oscillates.

If we continue in the direction which led to Theorem 3.2 with simpler equation

$$\begin{cases} [x(t)-px(t-\tau)]'+q(t)h(x(t-\sigma))=0,\ t\notin S \\ \Delta[x(t_k)-px(t_k-\tau)]+q_k h(x(t_k-\sigma))=0,\ \forall t_k\in S \end{cases} \tag{3.5}$$

where, $t, t_k\ge t_0$, we obtain the following result.

Theorem 3.3: Assume that $p\in(0, 1)$, $\tau\in(0, \infty)$, $q\in PC((t_0, \infty), R_+)$, $\sigma\in C,(t_0, \infty), R_+)$,

$$\sigma_0 \equiv \sup_{t \geq t_0} \sigma(t) < +\infty, \quad q_0 \equiv \sup_{t \geq t_0} q(t) < +\infty$$

and that

$$\liminf_{t \to +\infty} \left\{ (q(t) + q_k) \left[\frac{\sigma(t)}{1-p} + \frac{p\tau}{(1-p)^2} \right] \right\} > e^{-1} \quad (3.6)$$

Then, every solution of Eq. 3.5 oscillates.

Proof: Assume, for the sake of contradiction that Eq. 3.5 has a finally positive solution x. Set

$$v(t) = x(t) - px(t-\tau)$$

Then, by Lemma 2.2, there is a $t_1 = t_0$ such that

$$0 < v(t) < x(t)$$

for $\forall t = t_1$ and

$$\begin{cases} v'(t) + q(t)x(t - \sigma(t)) = 0, & t \notin S,\ t \geq t_1 - \xi \\ \Delta v(t_k) + q_k x(t - \sigma(t_k)) = 0, \forall t_k \in S \end{cases} \quad (3.7)$$

where,

$$\xi = \max\{\tau, \sigma_0\}$$

Observe that

$$x(t) = v(t) + px(t-\tau)$$

for, $t \geq t_1 - \xi$ and by induction, for $n \geq 1$, we find

$$x(t) = \sum_{\ell=0}^{n-1} p^\ell v(t - \ell\tau) + p^n x(t - n\tau),\ t \geq t_1 + (n-1)\tau - \xi$$

From this and Eq. 3.7, we see that the inequality

$$\begin{cases} v'(t) + q(t) \sum_{\ell=0}^{n-1} p^\ell v(t - \sigma(t) - \ell\tau) \leq 0, & \text{for } n \geq 1,\ t \notin S \\ \Delta v(t_k) + q_k \sum_{\ell=0}^{n-1} p^\ell v(t_k - \sigma(t_k) - \ell\tau) \leq 0, & n \geq 1,\ \forall t_k \in S \end{cases}$$

has a finally positive solution v. Hence, by Corollary 2.1, the equation

$$\begin{cases} u'(t) + q(t) \sum_{\ell=0}^{n-1} p^\ell u(t - \sigma(t) - \ell\tau) = 0, & \text{for } n \geq 1,\ t \notin S \\ \Delta u(t_k) + q_k \sum_{\ell=0}^{n-1} p^\ell u(t_k - \sigma(t_k) - \ell\tau) = 0, & n \geq 1,\ \forall t_k \in S \end{cases}$$

also has a finally positive solution. It follows that for every $n \geq 1$,

$$\liminf_{t \to +\infty} \left\{ (q(t) + q_k) \sum_{\ell=0}^{n-1} p^\ell [\sigma(t) + \ell\tau] \right\} \leq e^{-1} \quad (3.8)$$

We compute the sums of the series

$$\sum_{\ell=0}^{\infty} p^\ell = \frac{1}{1-p} \text{ and } \sum_{\ell=0}^{\infty} \ell p^\ell = \frac{p}{(1-p)^2}$$

Condition (3.8) therefore suggests that

$$\liminf_{t \to +\infty} \left\{ (q(t) + q_k) \left[\frac{\sigma(t)}{1-p} - \frac{p\tau}{(1-p)^2} \right] \right\} \leq e^{-1}$$

which, contradicts inequality (3.6) and completes the proof of Theorem 3.3.

CONCLUSION

From the results obtained, it is obvious that the anticipated oscillatory conditions depend not only on the coefficients and delays of the given equation, but also on the coefficients of the impulsive conditions. Their explicit nature and what is more, the ease of their verification are evidence enough of the advantage of these conditions over attempts to determine them via characteristic roots.

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