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# A Mathematical Model of HIV/AIDS Transmission Dynamics Considering Counselling and Antiretroviral Therapy

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**Abstract:** A mathematical model of HIV/AIDS transmission dynamics is proposed considering counselling and Antiretroviral Therapy (ART) as major means of control of infection. Threshold conditions are derived, in terms of the given model parameters, for the existence and stability of the disease-free and endemic equilibrium states of the model, as well as the proportion of infected people to receive ART. Analytical and numerical results obtained indicate that ART and counselling could be effective methods in the control and eradication of HIV/AIDS.

**Key words:** HIV/AIDS, ART, mathematical model, threshold conditions, numerical results, eradication

## INTRODUCTION

A major method, apart from the use of the condom, in the control of HIV/AIDS, is Antiretroviral Therapy (ART). By this approach, HIV positives are detected and placed on antiretroviral drugs. Generally, there are public awareness campaigns, which are intended to educate the general public on the spread of HIV and how to control it. Members of the public are encouraged to go for tests in order to determine their HIV/AIDS status so as to benefit from ART. ART does not cure HIV infection, it only boosts the immune system of infected people against secondary infections, thereby prolonging their life span. HIV positives are also detected through random screening and contact tracing.

Here, we propose a mathematical model of the dynamics of HIV, considering counseling and ART. The population is partitioned into three compartments of susceptible S(t), infected I(t) and removed R(t). A susceptible is an individual that is yet to be infected, but is open to infection as he or she interacts with members of the I-class. An infected individual is one, who has contracted HIV and is at some stage of infection. A removed individual is one that is confirmed to be HIV positive, counselledand is receiving ART.

It is assumed that the recruitment into the S-class is only through birth, at a rate b and it is proportional to the total population N(t) = S(t) + I(t) + R(t) at time t. Death is explicit in the model and it occurs in all classes at a constant rate $\mu$. However, there is an additional death rate $\alpha_0$ in the I and R-classes due to infection. There is a maximum period of time, T after infection, which a member in class I must leave the class through death. The death rate in the R-class is therefore, given by $\alpha = \alpha_0 e^{-kT}$, where k is the efficacy of the antiretroviral drug. The higher the value of k the smaller the value of $\alpha$ and vice versa. Clearly $\alpha<\alpha_0$ and $\alpha = \alpha_0$ when k = 0 (i.e., no ART).

The recruitment into the R-class from the I-class depends on the effectiveness of public campaign or counselling and this is done at a rate $\sigma$. $\sigma$ can also be referred to as the treatment rate. We ignore vertical transmission and age structure in the formulation. An age-structured formulation of a similar model has been proposed by Akinwande (2006), although not quite a SIR model as ours. Mathematical models to investigate the effect of treatment and vaccination on the spread of HIV/AIDS can be found, for example, in Kaosimore and Lungu (2004), Yang and Ferreira (1999), Hsu-Schmitz (2000), Swanson *et al.* (1994). Models for the control of HIV using the condom can be found, for example in Hsieh and Velasco-Hernandez (1995). Hsieh (1996), Mastro and Limpakarnjanarat (1995), Kimbir and Aboiyar (2003) and Kimbir *et al.* (2006).

## FORMULATION OF THE MODEL EQUATIONS

The following diagram will be found useful in formulating the model Equations.

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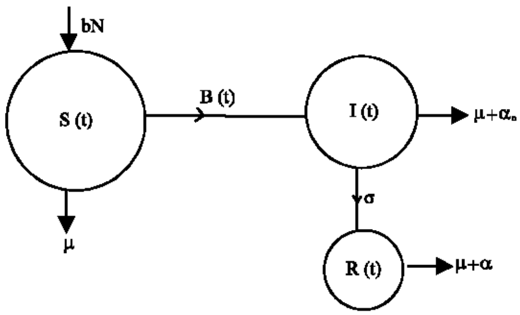


Fig. 1: A flow diagram of the transmission of HIV considering counselling and ART

From the assumptions in the study and the above diagram (Fig. 1) the following model Equations are derived.

$$\frac{dS}{dt} = bN - B(t)S - \mu S, \tag{1}$$

$$\frac{dI}{dt} = B(t)S - (\mu + \alpha_0 + \sigma)I, \tag{2}$$

$$\frac{dR}{dt} = \sigma I - (\mu + \alpha)R \tag{3}$$

where:

$$N(t) = S(t) + I(t) + R(t) \tag{4}$$

and

$$\alpha = \alpha_0 e^{-kT} \tag{5}$$

The incidence rate B(t) at time t is given as in Hsieh (1996), namely

$$B(t) = \frac{c\beta I + c'\beta' R}{N} \tag{6}$$

where, $\beta$ is the probability of transmission by an individual in class I and $\beta'$ is the probability of transmission by an individual in class R, c and c' are, respectively, the average number of sexual partners per unit time for individuals in class I and R. B is the reproduction rate of the population, $\alpha_0$, $\alpha$ and $\sigma$ are as defined in section 1. $c\beta$ and $c'\beta'$ are therefore, the net transmission rates for the classes I and R, respectively. As a result of counselling, it is assumed that $c'\beta' < c\beta$. For ease of reference we redefine the model parameters in the following Table 1.
Adding Eq. (1-3), we have

$$\frac{dN}{dt} = (b - \mu)N - a_0 I - \alpha R \tag{7}$$

Table 1: Model parameters

| | | |
|---|---|---|
| S(t) | = | Number of susceptible at time t |
| I(t) | = | Number of infected at time t |
| R(t) | = | Number of infected receiving ART at time t |
| b | = | Population birth rate |
| $\mu$ | = | Population death rate |
| $\alpha_0$ | = | Population death rate of infected not receiving ART |
| $\alpha$ | = | Population death rate of infected receiving ART |
| T | = | Maximum lifespan after infection |
| k | = | Efficacy of ART per unit time |
| c | = | Average number of sexual partners of members of class I |
| c | = | Average number of sexual partners of members of class R |
| $\beta$ | = | Probability of transmission by members of class I |
| $\beta'$ | = | Probability of transmission by members of class R |
| $\sigma$ | = | Proportion of infected receiving ART per unit time |

Let,

$$s = \frac{S}{N},\ i = \frac{I}{N} \text{ and } r\ \frac{R}{N},$$

then we have s = 1 - i - r and the governing Equations of the model, in proportions i and r, are given below.

$$\frac{di}{dt} = (c\beta i + c'\beta' r)(1 - i - r) \\ -(\alpha_0 + b + \sigma)i + \alpha i r + \alpha_0 i^2 \tag{8}$$

$$\frac{dr}{dt} = \sigma i - (\alpha + b)r + \alpha_0 i r + \alpha r^2 \tag{9}$$

The Equations in proportions have biological meaning as they define prevalence of infection.

## EXISTENCE AND STABILITY OF EQUILIBRIUM STATES

It is easy to see that (0, 0), is an equilibrium state of the model Eq. (8) and (9).

The Jacobian matrix $J_0$ associated with the equilibrium state (0, 0) is given by:

$$J_0 = \begin{pmatrix} c\beta - (\alpha_0 + b + \sigma) & c'\beta' \\ \sigma & -(\alpha + b) \end{pmatrix}$$

Let,

$$R = \frac{c\beta}{\alpha_0 + b + \sigma}$$

and define D = {(I, r): i>0, r>0, i + r = 1}, then we have the following result. Shown in Fig. 2.

**Theorem:** Given $\alpha$, $\alpha_0$, b, $\sigma$, $c\beta$, $c'\beta'>0$. If $\alpha + b > \alpha_0$ and $0<R<1$, then there exists a Disease-Free Equilibrium state

(DFE), (0, 0), which is Locally and Asymptotically Stable (LAS), otherwise there exists an endemic state (i*, r*), which is LAS in D - {(0, 0)}.

**Proof:** We see from the hypotheses of the theorem that

$$TrJ_0 = (\alpha_0 + b + \sigma)(R - 1) - (\alpha + b) < 0$$

$$detJ_0 > -(\alpha_0 + b + \sigma)^2 (R - 1) > 0.$$

Therefore, the disease-free state is locally and asymptotically stable. We know that the DFE is unstable if the condition R<1 does not hold, that is if $R \geq 1$. In this case, we only need to show that the region D is invariant, containing no periodic solutions of the system Eq. (8) and (9), so that all solutions tend to the endemic equilibrium state (i*, r*).

First, we shall show that D is an invariant region. We do this by showing, as in Beltrami (1989), that the inner product of the vector field defined by Eq. (8-9) with the inward normal to D is non-negative.

Let, $f_1(i, r)$ and $f_2(i, r)$ denote, respectively, the rhss of Eq. (8-9). Going back to Fig. 2, we see that the inward normal to the i-axis is (0, 1) therefore,

$$(0, 1)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_2 = (\sigma i - (\alpha + b))r + \alpha_0 ri + \alpha r^2 > 0$$

(since, r = 0 on this axis). Next the inward normal to the r axis is (1, 0), so that

$$(1, 0)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = f_1 = c'\beta' r(1 - r) > 0$$

(since, i = 0 on this axis and r<1). Finally, on the line i + r = 1, we have

$$\frac{1}{\sqrt{2}}(-1, -1)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} > 0,$$

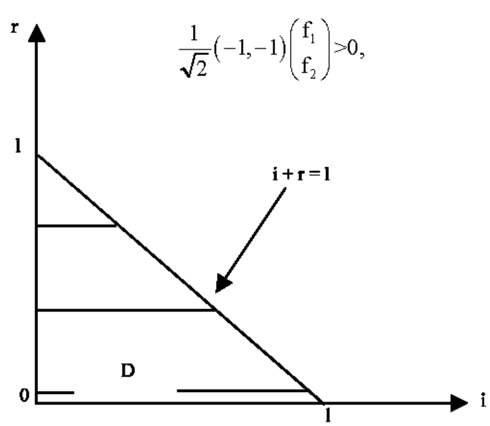


Fig. 2: The region D in R2

using similar arguments. Thus, we have proved that D is invariant. It remains to prove, using the Bendixon-Dulac criterion, as in Hsieh (1996), that there are no periodic solutions of the system (8) and (9). Let, g = 1/ir, then we have

$$\frac{\partial}{\partial i}(gf_1) + \frac{\partial}{\partial r}(gf_2) < -\left(\frac{\sigma}{r^2} + \frac{\alpha}{i}\right) < 0$$

Therefore, there are no periodic solutions of the system in D. Hence, the proof.

## RESULTS AND DISCUSSION

In this study, we formulated and studied a mathematical model for the transmission of HIV/AIDS considering counselling and Antiretroviral Therapy (ART). The model parameters are shown in Table 1. The model Equations are derived with the help of a flow diagram in Fig. 1.

The main result of the study, is found in theorem 1, where threshold conditions are given for the stability of the disease-free and the endemic equilibrium states of the model. Whereas the condition $\alpha + b > \alpha_0$ holds vacuously, the number

$$R = \frac{c\beta}{\alpha_0 + b + \sigma}$$

may not always be >1 From the rhs of the expression for R, we see that increasing the value of sigma (i.e., increasing the treatment rate) reduces the value of R below 1. Similarly, reducing the value of c or $\beta$ may achieve the same purpose. Thus, for an effective ART programme, it may be necessary to also reduce the transmission probability and the average number of sexual partners of the infected individuals. These can be done through counselling and education. From the expression

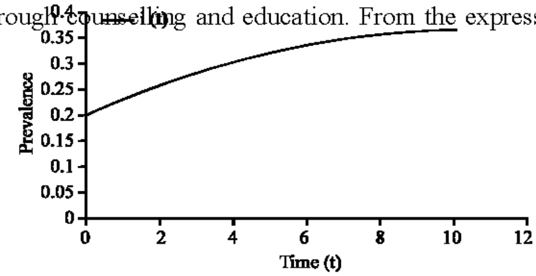


Fig. 3: Prevalence of Infection without any intervention

($\sigma$ = 0). Other parameter values are: b = 0.5, $\alpha_0$ = 0.2, T = 10, k = 0, c = 1, c' = 0, = 1, $\beta'$ = 0

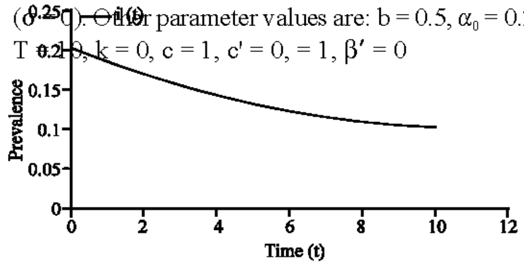


Fig. 4: Prevalence of infection with low treatment rate ($\sigma$ = 0.2). Other parameter values are: b = 0.5, $\alpha_0$ = 0.2, T = 10, k = 0, c = 1, c' = 0, $\beta$ = 1, $\beta'$ = 0

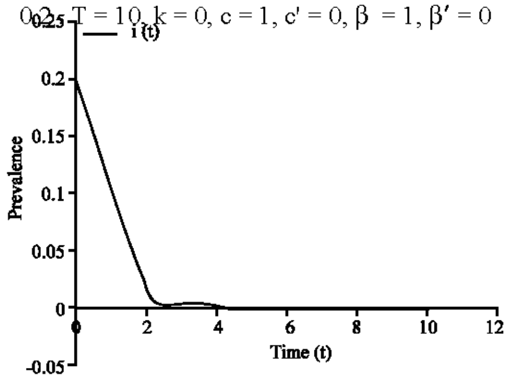


Fig. 5: Prevalence of infection with high treatment rate ($\sigma$ = 0.8). Other parameter values are: b = 0.5, $\alpha_0$ = 0.2 , T = 10, k = 0, c = 1, c' = 0, $\beta$ = 1, $\beta'$ = 0.

R<1, we see that the minimum proportion $\sigma$ of infected individuals to receive ART is $c\beta - (\alpha_0 + b)$.

Numerical examples, using hypothetical data, satisfying the inequality $c\beta>(\alpha_0+ b)$ give the following results. Figure 3 shows an increasing prevalence in the absence of ART (i.e., $\sigma$ = 0). Figure 4 shows the prevalence of infection when the treatment rate is low (i.e., $\sigma$ = 0.2), while Fig. 5 shows prevalence of I nfection when the treatment rate is high (i.e., $\sigma$ = 0.8). Hence, this study confirms that counselling and ART could be useful methods for the control and eradication of HIV/AIDS.

## REFERENCES

Akinwande, N.I., 2006. A mathematical model of the dynamics of the HIV/AIDS disease pandemic. J. Nig. Math. Soc., 25: 99-108.

Beltrami, E., 1989. Mathematics for dynamic modeling. Academic Press. N.Y.

Hsieh, H.Y. and J.X. Velasco-Hernandez, 1995. Community treatment of HIV-1 initial stage and asymptotic analysis. Biosystems, 25: 75-81.

Hsieh, H.Y., 1996. A two sex model for treatment of AIDS and behaviour change in a population of varying size. IMA. J. Math. Applied Bio. Med., 13: 151-173.

Hsu-Schmitz, S.F., 2000. Effects of treatment or/and vaccination on HIV transmission in homosexuals with genetic heterogeneity. Math. Biosci., 167: 1-18.

Kaosimore, M. and E.M. Lungu, 2004. Effects of vaccination and treatment on the spread of HIV/AIDS. J. Biol. Syst., 12 (4): 399-417.

Kimbir, A.R. and T. Aboiyar, 2003. A mathematical model for the prevention of HIV/AIDS in a varying population. J. Nig. Math. Soc., 22: 43-55.

Kimbir, A.R., S. Musa and E.B. Bassey, 2006. On a two-sex mathematical model for the prevention of HIV/AIDS in a varying population. ABACUS J. Math. Asoc. Nig., 33 (201): 1-13.

Mastro, T.D. and K. Limpakarnjanarat, 1995. Condom use in Thailand: How much is it slowing down the HIV/AIDS epidemic? AIDS., 9: 523-525.

Swanson, C.E., B. Tindall and D.A. Cooper, 1994. Efficacy of Zidovudine treatment in homosexual men with AIDS-related complex. Factors influencing development of AIDS, sexual and drug tolerance-AIDS., 9: 625-634.

Yang, H.M. and W.C. Ferreira, 1999. A population model applied to HIV transmission, considering protection and treatment. IMA J. Maths. Applied Med. Biol., 16: 237-259.