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On the Existence of Solution of Differential Equation of Fractional Order

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Abstract: We are concerned with the solution of the differential equations of the form:

$$\frac{d^{\alpha}y}{dx^{\alpha}}(x)+\lambda y=f(x) \tag{1}$$

where, $\alpha = m/n$, $n\neq 0$. Equations of this type arise in the generalized viscoelastic constitutive equations and in fractional Brownian motion. Numerical methods for solution of Eq. (1) are well established, particularly for $0<m/n<1$. In this study, we present an Adomian decomposition method for the solution of prototype Eq. (1), with $1<m/n<2$.

Key words: Differential equation, ADM, fractional order differential equations

INTRODUCTION

Fractional calculus is the branch of calculus that generalizes the derivative of a function to a non-integer order, allowing calculation such as deriving a function to ½ order. The usual formulation of the fractional derivative given in standard references such as (Oldham and Spanier, 1974; Sanko *et al*., 1993), is the Riemann Liouville definition. In application of the fractional derivative suggested by (Caputo, 1999). For the sake of convenience, we give definitions of fractional integral and fractional derivative introduced by Riemann-Lioville as discussed in Gorenflo and Mainardi (1996).

Definition: The fractional derivative of a function y defined on the interval (O, T) at time t ∈ (O, T) is given by the convolution integral:

$$D^{\alpha}y=\frac{1}{\Gamma(a-n)}\int_0^t\frac{y^n(x)dx}{(t-x)^{\alpha+1-n}}$$

where:

$$n \in N^t \text{ and } \alpha \in (n, n+1)$$

The α order Caputo derivative of x^{μ} is given by:

$$D^{\alpha}x^{\mu}=\frac{\Gamma(\mu+1)\,x^{\mu-\alpha}}{\Gamma(\mu+1-\alpha)},\ x>0$$

which can be applied to nth order differential equation (Diethelm and Ford, 1999).

Adomian Decomposition Method (ADM): Consider a general nonlinear equation of the form:

$$Lu + Ru + Nu = g \tag{2}$$

where:

L = Is the highest order derivative, which is invertible.
R = The linear differential operator of less order than.
L = Nu represents the nonlinear terms.
g = Is the source term.

Applying L^{-1} to Eq. (2):

$$U = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{3}$$

For nonlinear differential equations, Nu = F (u) is represented by Adomian polynomial:

$$F(u)=\sum_{m=0}^{\alpha} Am \tag{4}$$

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Thus, the polynomial depend on the nonlinearity:

$$\begin{aligned}
A_0 &= F(U_0) \\
A_1 &= U_1 F^1(U_0) \\
A_2 &= U_2\, F^1(U_0) + \frac{1}{2^1} U_1^{\,2}\, F^{11}(U_0) \\
A_3 &= U_3\, F^1(U_0) + U_1\, U_0 F^{11}(U_0) \\
&+ \frac{1}{3^1} U_1^{\,3} F^{111}(U_0) \\
&\vdots
\end{aligned} \tag{5}$$

and generally, from :

$$A_m = \sum_{V=1}^{m} c(V,m) f^{v}(U_0) \tag{6}$$

Numerical example: Consider:

$$\frac{dy}{dx} + \frac{dy^{3/2}}{dx^{3/2}} - y = 0 \tag{7}$$

$x>0$, $y^k(0) = 0$,
$(n-1)<3/2<n$, $k = 0, 1 \ldots n-1$

By ADM, we can write:

$$L^{-1}\left(\frac{dy}{dx}\right) + L^{-1}\left(\frac{dy^{3/2}}{dx^{3/2}}\right) - L^{-1}(y) = L^{-1}(0)$$

therefore,

$$y = c - \frac{dy^{1/2}}{dx^{1/2}} + \frac{d^{-1}y}{dx^{-1}}$$

let, $y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots + y_n$ (8)

be a solution of Eq. (7):

$$\begin{aligned}
y_0(x) &= c \\
y_1(x) &= \frac{d^{-1}\, y_0(x)}{dx^{-1}} - \frac{d^{1/2} y_0(x)}{dx^{1/2}} \\
y_2(x) &= \frac{d^{-1} y_1(x)}{dx^{-1}} - \frac{d^{1/2} y_1(x).}{dx^{1/2}} \\
y_3(x) &= \frac{d^{-1} y_2(x)}{dx^{-1}} - \frac{d^{1/2} y_2(x)}{dx^{1/2}} \\
&\vdots
\end{aligned}$$

which implies that:

$$\begin{aligned}
y_1(x) &= \left(x - \frac{x^{-1/2}}{5\sqrt{\pi}}\right)c \\
y_2(x) &= \left(\frac{x^2}{x} - \frac{4x^{1/2}}{\sqrt{\pi}} - x^{-1}\right)c \\
y_3(x) &= \left(\frac{3}{2}x^3 - \frac{8x^{3/2}}{3\sqrt{\pi}} - x^{-2} - \frac{4x^{3/2}}{3\sqrt{\pi}} + 2 - 2\sqrt{\pi}x^{-3/2}\right)c \\
y_4(x) &= \left(\begin{aligned}&\frac{3x^4}{8} + 2x - \frac{257}{20\sqrt{\pi}}x^{5/2} + x^{-1} + \frac{2x}{\pi} + \frac{3x^{-5/2}}{\sqrt{\pi}} \\ &+\left(4\sqrt{\pi} + 4\pi - \frac{2}{\sqrt{\pi}}\right)x^{-1/2}\end{aligned}\right)c \\
&\vdots
\end{aligned}$$

$$\therefore y(x) = \left(\begin{aligned}&3 + 3x + \frac{2x}{\pi} + \frac{x^2}{2} - \frac{1}{x^2} + \frac{3x^3}{2} + \frac{3x^4}{8} \\ &+\left(4\sqrt{\pi} + 4\pi - \frac{3}{\sqrt{\pi}}\right)x^{-1/2} \\ &-\frac{4x^{1/2}}{\sqrt{\pi}} - 2\sqrt{\pi}x^{-3/2} - \frac{4}{\sqrt{\pi}}x^{3/2} - \frac{257}{20\sqrt{\pi}}x^{5/2} \\ &+\frac{3x^{-5/2}}{\sqrt{\pi}} + \ldots\ldots + \ldots\end{aligned}\right) \tag{9}$$

where, c is an arbitrary constant.

CONVERSION OF FRACTIONAL ORDER TO INTEGER ORDER DIFFERENTIAL EQUATION

From Eq. (7), applying $L^{-1/2}$ to both sides, we have:

$$\frac{d^{1/2}y}{dx^{1/2}} + \frac{dy}{dx} - \frac{d^{-1/2}y}{dx^{-1/2}} = Cx^{-1/2} \tag{10}$$

Differentiating and applying $L^{1/2}$ to both sides, we have:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = c\sqrt{\pi}x^{-2}$$

Therefore,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = kx^{-2} \tag{11}$$

where, k is a constant.

It can be shown that Eq. (11) has the same solution with Eq. (7).

CONCLUSION

In this study, ADM has been successfully used to solve fractional order differential equation of order greater than 1.

The results obtained are compared with integer-order differential equation, which shows agreement.

The result shows that Adomian decomposition method is efficient and reliable methods for linear problems.

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