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Inventory System Exposed to Calamities with SCBZ Arrival Property

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Abstract: This study is devoted to the analysis of (s, S) inventory system where the demand process is assumed to a combination of single and bulk demand for entire inventory where the rates of demand have the SCBZ property. The lead times and intervals of time between successive demands are i.i.d random variables. We discuss the exponential case of 2 models. In model 1, unit demand rates are varying and in model 2, bulk demand rates are varying. Steady state probability vector of inventory levels is obtained using NEUTS matrix method by Nutes in 1980. Numerical examples are also presented.

Key words: Inventory systems, SCBZ property, renewal theory, arrival property

INTRODUCTION

Single commodity inventory problem of (s, S) type was discussed by several researchers. Daniel and Ramanarayanan (1988) discussed (s, S) inventory with random lead time and unit demand. Ramanarayanan and Jacob (1998) studied this system with bulk demand. In all these models, either unit demand or bulk demand was taken up for discussion separately. Chenniappan and Ramanarayanan (1995) treated (s, S) inventory system exposed to calamities. In this study, the demand process is assumed to be a combination of single and bulk demand for entire inventory where the rates of demand have the SCBZ (Setting Clock Bock to Zero) property.

For a detailed study of the SCBZ property, one may refer to Raja Rao (1998). According to this property the probability distribution of random variable has a parametric change after a truncation point. In models considered here the parametric changes occur for unit demand or bulk demand rates after an exponential time.

Considering exponential distribution arrival, calamity and lead time, 2 models are studied in this study using block partitioning of the infinitesimal generator of the underlying continuous time Markov chain of the systems. The steady state probability vectors of the two models are interesting. Numerical examples are also presented. In model 1 arrival rate changes after an exponential time and the calamity rate is constant. When the inventory contains perishable items one may like to charge the arrival rate of demands to improve sales. In model 2 arrival rate for unit demand is constant and bulk demand rate charges after an exponential time to make profits or clear stocks.

MODEL-1: VARYING UNIT DEMAND RATES

The following are the assumptions of the model.

- The maximum capacity of the inventory is S and ordering level is s (S-s > s).
- The arrival rate of unit d emand is λ_1. If it does not occur within an exponential time with parameter c the arrival rate changes to λ_2. Immediately upon arrival, the rate becomes λ_1.
- The rate of occurrence of calamity is α.
- When the inventory falls to the level s from above an order for S-s units are placed. The lead time distribution for such an order is exponential with parameter μ.
- When a calamity occurs an order is placed for S units, the lead time distribution is exponential with rate β and order pending if any is cancelled.

The inter arrival time distribution of unit demands may be derived as follows. Let Y be the time between two consecutive unit demands. Let its pdf be h(y).

Considering the truncation point τ_0 in which the parameter changes we note

$$h(y) = \begin{cases} \lambda_1 e^{-\lambda_1 y} & \text{if } y \le \tau_0 \\ \lambda_2 e^{-\lambda_2 y} e^{\tau_0(\lambda_2-\lambda_1)} & \text{if } y > \tau_0 \end{cases} \quad (1)$$

Noting the truncation point τ_0 itself is a random variable with pdf $ce^{-c\tau_0}$, we find

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$$h(y)=\lambda_1 e^{-\lambda_1 y} e^{-cy} + \lambda_2 e^{-\lambda_2 y} \int_0^y e^{\tau_0(\lambda_2-\lambda_1)}\, c e^{-c\tau_0} d\tau_0$$

which reduces to

$$h(y)=\frac{(\lambda_1-\lambda_2)(c+\lambda_1)}{(c+\lambda_1-\lambda_2)} e^{-y(\lambda_1+c)} + \frac{c\lambda_2 e^{-\lambda_2 y}}{(c+\lambda_1-\lambda_2)} \qquad (2)$$

and the distribution function of time between unit demands with varying parameter after exponential time is given by

$$H(y) = 1 - p\, e^{-y(\lambda_1+c)} - q e^{-\lambda_2 y} \qquad (3)$$

Where:

$$p=\frac{\lambda_1-\lambda_2}{c+\lambda_1-\lambda_2}$$

and

$$p=\frac{c}{c+\lambda_1-\lambda_2}$$

Also we note that p + q = 1.

For Markovian models it is advantageous to set up the infinitesimal generator for finding probabilities. The arrival process of unit demands has two phases with rates λ_1 and λ_2. The state of the system may be written as follows.

$$S = \{(i, j) \colon 1 \le i \le S,\ j = 1,2\} \cup \{0\} \cup \{0^*\} \qquad (4)$$

The inventory is in the state (i, j) when i units are in the inventory and the unit arrival rate is λ_j for $1 \le i \le S$ and j = 1 or 2. The inventory is in state 0 when no unit is in the inventory and lead time rate is μ to supply S-s units. The inventory is in state 0^* when no unit is in the inventory due to a calamity and the lead time rate is β to supply S units. The infinite generator Q of the continuous time Markov chain is given by:

$$Q=\begin{array}{c} \\ \underline{S} \\ \underline{S-1} \\ \underline{S-2} \\ \vdots \\ \underline{S-s+1} \\ \underline{S-s} \\ \vdots \\ \underline{s+1} \\ \underline{s} \\ \underline{s-1} \\ \underline{s-2} \\ \vdots \\ \underline{3} \\ \underline{2} \\ \underline{1} \\ 0 \\ 0^* \end{array}
\begin{array}{ccccccccccccccccc}
\underline{S} & \underline{S-1} & \underline{S-2} & \cdots & \underline{S-s+1} & \underline{S-s} & \cdots & \underline{s+1} & \underline{s} & \underline{s-1} & \underline{s-2} & \cdots & \underline{3} & \underline{2} & \underline{1} & 0 & 0^* \\
T & A & & & & & & & & & & & & & & & \underline{\alpha'} \\
 & T & A & & & & & & & & & & & & & & \underline{\alpha'} \\
 & & \ddots & \ddots & & & & & & & & & & & & & \vdots \\
 & & & \ddots & \ddots & & & & & & & & & & & & \underline{\alpha'} \\
 & & & & T & A & & & & & & & & & & & \underline{\alpha'} \\
 & & & & & T & \ddots & & & & & & & & & & \vdots \\
 & & & & & & \ddots & A & & & & & & & & & \vdots \\
 & & & & & & & T & A & & & & & & & & \underline{\alpha'} \\
\mu I & & & & & & & & T' & A & & & & & & & \underline{\alpha'} \\
 & \mu I & & & & & & & & T' & A & & & & & & \underline{\alpha'} \\
 & & \mu I & & & & & & & & T' & A & & & & & \underline{\alpha'} \\
 & & & & & & & & & & & T' & \ddots & & & & \vdots \\
 & & & \ddots & & & & & & & & & \ddots & \ddots & & & \vdots \\
 & & & & & & & & & & & & & \ddots & \ddots & & \vdots \\
 & & & & \mu I & & & & & & & & & & T' & A' & \underline{\alpha'} \\
 & & & & & (\mu,0) & & & & & & & & & & -\mu-\alpha & \alpha \\
(\beta,0) & & & & & & & & & & & & & & & & -\beta
\end{array}$$

(5)

The sub matrices inside the infinitesimal generator Q are given as follows

$$T=\begin{bmatrix}\lambda_1-\alpha-c & c \\ 0 & \lambda_2-\alpha\end{bmatrix}, A=\begin{bmatrix}\lambda_1 & 0 \\ \lambda_2 & 0\end{bmatrix}, \underline{\alpha'}=\begin{bmatrix}\alpha \\ \alpha\end{bmatrix}, A'=\begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}, T'=\begin{bmatrix}-\lambda_1-\alpha-c-\mu & c \\ 0 & \lambda_2-\alpha-\mu\end{bmatrix} \qquad (6)$$

The matrix Q is of order 2(S + 1) and all the unmarked entries are zero. Let $\underline{\Pi}$ be the vector of steady state probabilities associated with Q satisfying

$$\underline{\Pi} Q = 0 \text{ and } \underline{\Pi}\,\underline{e} = 1 \tag{7}$$

Where:

$$\underline{\Pi} = \left(\underline{\Pi}_{S,} \underline{\Pi}_{S-1}, \cdots \underline{\Pi}_1, \Pi_0, \Pi_{0*}\right)$$

and

$$\underline{e} = (1, 1, 1, \cdots, 1)^t \cdot$$

We note from Eq. 7

$$\underline{\Pi}_s T' + \underline{\Pi}_{s+1} A = 0$$

which implies

$$\underline{\Pi}_s = \underline{\Pi}_{s+1} A\, (-T')^{-1} \cdot$$

Similarly, we may note

$$\underline{\Pi}_{s-j} = \underline{\Pi}_{s+1} \left[A\, (-T')^{-1} \right]^{j+1} \tag{8}$$

for $j = 0, 1, 2, \cdots, s-1$.

Now

$$\Pi_0 (-\mu - \alpha) + \underline{\Pi}_1 A' = 0$$

gives

$$\Pi_0 = \underline{\Pi}_{s+1} \left[A\, (-T')^{-1} \right]^s \frac{A'}{\mu + \alpha}. \tag{9}$$

Now

$$\underline{\Pi}_S T + \underline{\Pi}_s \mu I + \Pi_{0^*} (\beta, 0) = 0$$

gives

$$\underline{\Pi}_S = \underline{\Pi}_{s+1}\; A (-T')^{-1}\, (\mu I)\, (-T)^{-1} + \Pi_{0^*}\; (\beta, 0)\, (-T)^{-1} \tag{10}$$

Further

$$\underline{\Pi}_S A + \underline{\Pi}_{S-1} T + \underline{\Pi}_{s-1} \mu I = 0$$

$$\underline{\Pi}_{S-1} = \underline{\Pi}_S A\, (-T)^{-1} + \underline{\Pi}_{s-1} (\mu I)(-T)^{-1}$$

$$\underline{\Pi}_{S-1} = \left[\underline{\Pi}_{s+1} A\, (-T')^{-1} (\mu I)\, (-T)^{-1} + \Pi_{0^*} (\beta, 0)(-T)^{-1} \right] A(-T)^{-1} + \underline{\Pi}_{s+1} \left[A(-T')^{-1} \right]^2 (\mu I)(-T)^{-1}.$$

On simplification

$$\underline{\Pi}_{S-1} = \underline{\Pi}_{s+1} A\, (-T')^{-1} (\mu I)\, (-T)^{-1} A(-T)^{-1} + \Pi_{0^*} (\beta, 0)(-T)^{-1} A(-T)^{-1} + \underline{\Pi}_{s+1} \left[A(-T')^{-1} \right]^2 (\mu I)(-T)^{-1}.$$

Also,

$$\underline{\Pi}_{S-1} A + \underline{\Pi}_{S-2} T + \underline{\Pi}_{s-2}\; \mu\; I = 0$$

which implies

$$\underline{\Pi}_{S-2} = \underline{\Pi}_{S-1} A\, (-T)^{-1} + \underline{\Pi}_{s-2}\, \mu\, I\, (-T)^{-1}$$

gives

$$\underline{\Pi}_{S-2} = \underline{\Pi}_{s+1} A\, (-T')^{-1} (\mu I)\, (-T)^{-1} \left[A(-T)^{-1} \right]^2 + \Pi_{0^*} (\beta, 0)(-T)^{-1} \left[A(-T)^{-1} \right]^2 + \underline{\Pi}_{s+1} \left[A(-T')^{-1} \right]^2 (\mu I)(-T)^{-1} A(-T)^{-1} + \underline{\Pi}_{s+1} \left[A(-T')^{-1} \right]^3 (\mu I)(-T)^{-1}.$$

On simplification,

$$\underline{\Pi}_{S-2} = \underline{\Pi}_{s+1} \left[\sum_{j=1}^{j=3} \left[A\, (-T')^{-1} \right]^j (\mu I)\, (-T)^{-1} \left[A(-T)^{-1} \right]^{3-j} \right] + \Pi_{0^*} (\beta, 0)(-T)^{-1} \left[A(-T)^{-1} \right]^2$$

Similarly we may find

$$\underline{\Pi}_{S-i} = \underline{\Pi}_{s+1}\left[\sum_{j=1}^{j=i+1} \left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{i+1-j}\right] + \Pi_{0^*}(\beta,0)(-T)^{-1}\left[A(-T)^{-1}\right]^{i}$$

for $1 \le i \le s-1$.

Now

$$\underline{\Pi}_{S-s+1}A + \underline{\Pi}_{S-s}T + \Pi_0(\mu,0) = 0$$

$$\underline{\Pi}_{S-s} = \underline{\Pi}_{S-s+1}A\ (-T)^{-1} + \Pi_0(\mu,0)(-T)^{-1}$$

$$\underline{\Pi}_{S-s} = \left\{\underline{\Pi}_{s+1}\left[\sum_{j=1}^{s} \left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-j}\right] + \Pi_{0^*}(\beta,0)(-T)^{-1}\left[A(-T)^{-1}\right]^{s-1}\right\}$$
$$\left[A(-T)^{-1}\right] + \underline{\Pi}_{s+1}\left[A\,(-T')^{-1}\right]^{s}\frac{A'}{\mu+\alpha}(\mu,0)(-T)^{-1}$$

$$\underline{\Pi}_{S-s} = \underline{\Pi}_{s+1}\left[\sum_{j=1}^{s} \left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-j}\right]\left[A(-T)^{-1}\right]$$
$$+\underline{\Pi}_{s+1}\left[A\,(-T')^{-1}\right]^{s}\frac{A'}{\mu+\alpha}(\mu,0)(-T)^{-1} + \Pi_{0^*}(\beta,0)(-T)^{-1}\left[A(-T)^{-1}\right]^{s} \quad (12)$$

Next,

$$\underline{\Pi}_{S-s}A + \underline{\Pi}_{S-s-1}T = 0$$

gives

$$\underline{\Pi}_{S-s-1} = \underline{\Pi}_{S-s}\left[A\ \ (-T)^{-1}\right]$$

We note that

$$\underline{\Pi}_{S-s-j} = \underline{\Pi}_{S-s}\left[A\ \ (-T)^{-1}\right]^{j}$$

for $j = 1, 2, \ldots,\ S - 2s - 2$

Therefore,

$$\underline{\Pi}_{S-s-j} = \underline{\Pi}_{s+1}\left[\sum_{k=1}^{s} \left[A\,(-T')^{-1}\right]^{k}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-k}\right]\left[A(-T)^{-1}\right]^{j+1}$$
$$+\underline{\Pi}_{s+1}\left[A\,(-T')^{-1}\right]^{s}\frac{A'}{\mu+\alpha}(\mu,0)(-T)^{-1}\left[A(-T)^{-1}\right]^{j} + \Pi_{0^*}(\beta,0)(-T)^{-1}\left[A(-T)^{-1}\right]^{s+j} \quad (13)$$

for $1 \le j \le S - s - 2$.

Also

$$\underline{\Pi}_{s}\,\underline{\alpha}' + \underline{\Pi}_{s-1}\,\underline{\alpha}' + \cdots + \underline{\Pi}_{s+1}\,\underline{\alpha} + \cdots + \underline{\Pi}_{1}\,\underline{\alpha} + \Pi_o\alpha + \Pi_{0^*}(-\beta) = 0$$

This gives

$$\Pi_{0*}\left[(\beta,0)(-T)^{-1}\sum_{j=0}^{S-2s-2}\left[A(-T)^{-1}\right]^{j}\underline{\alpha}'-\beta\right]$$
$$+\underline{\Pi}_{s+1}\left\{\left[\left[A\,(-T')^{-1}(\mu I)\,(-T)^{-1}+\sum_{i=1}^{s-1}\sum_{j=1}^{i+1}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{i+1-j}\right.\right.\right.$$
$$+\left(\sum_{j=1}^{s}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-j}\right)\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j+1}\right)$$
$$+\left[A(-T)^{-1}\right]^{s}\left(\frac{A'}{\mu+\alpha}\right)(\mu,0)(-T)^{-1}\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j}\right)+1$$
$$\left.\left.+\sum_{j=0}^{s-1}\left[A\,(-T')^{-1}\right]^{j+1}\right]\underline{\alpha}'+\left[A(-T)^{-1}\right]^{s}\left(\frac{A'\alpha}{\mu+\alpha}\right)\right\}\underline{\alpha}'=0$$

$$\Pi_{0*}=\underline{\Pi}_{s+1}\left\{\left[\left[A\,(-T')^{-1}(\mu I)\,(-T)^{-1}+\sum_{i=1}^{s-1}\sum_{j=1}^{i+1}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{i+1-j}\right.\right.\right.$$
$$+\left(\sum_{j=1}^{s}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-j}\right)\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j+1}\right)$$
$$+\left[A(-T')^{-1}\right]^{s}\left(\frac{A'}{\mu+\alpha}\right)(\mu,0)(-T)^{-1}\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j}\right)+1 \tag{14}$$
$$\left.\left.+\sum_{j=0}^{s-1}\left[A\,(-T')^{-1}\right]^{j+1}\right]\underline{\alpha}'+\left[A(-T')^{-1}\right]^{s}\left(\frac{A'\alpha}{\mu+\alpha}\right)\right\}\left[\beta-(\beta,0)(-T)^{-1}\sum_{j=0}^{S-2s-2}[A(-T)^{-1}]^{j}\,\underline{\alpha}'\right]^{-1}$$

$\underline{\Pi}_{s+1}$ is now calculated using the total law of probability $\underline{\Pi}_{S}\,\underline{e}+\underline{\Pi}_{S-1}\underline{e}+\cdots+\underline{\Pi}_{1}\underline{e}+\Pi_0+\Pi_{0^*}=1$. This gives using Eq. 8-14

$$\underline{\Pi}_{s+1}.\underline{\alpha}=1 \tag{15}$$

and

$$\underline{\Pi}_{s+1}=\frac{\underline{\alpha}^{t}}{|\,\underline{\alpha}\,|^{2}} \tag{16}$$

Where:

$$\underline{a}=\left\{\left[\left[A\,(-T')^{-1}(\mu I)\,(-T)^{-1}+\sum_{i=1}^{s-1}\sum_{j=1}^{i+1}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{i+1-j}\right.\right.\right.$$
$$+\left(\sum_{j=1}^{s}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-j}\right)\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j+1}\right)$$
$$+\left[A(-T)^{-1}\right]^{s}\left(\frac{A'}{\mu+\alpha}\right)(\mu,0)(-T)^{-1}\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j}\right)+1$$
$$\left.\left.+\sum_{j=0}^{s-1}\left[A\,(-T')^{-1}\right]^{j+1}\right]\underline{\alpha}'+\left[A(-T)^{-1}\right]^{s}\left(\frac{A'\alpha}{\mu+\alpha}\right)\right\}\left[\beta-(\beta,0)(-T)^{-1}\sum_{j=0}^{S-2s-2}[A(-T)^{-1}]^{j}\,\underline{\alpha}'\right]^{-1}$$
$$\left[\beta-(\beta,0)(-T)^{-1}\sum_{j=0}^{S-2s-2}[A(-T)^{-1}]^{j}\,\underline{e}+1\right]$$

$$+\left\{\left[A\,(-T')^{-1}(\mu I)\,(-T)^{-1}+\sum_{i=1}^{s-1}\sum_{j=1}^{i+1}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{i+1-j}\right.\right.$$
$$+\left(\sum_{j=1}^{s}\left[A\,(-T')^{-1}\right]^{j}(\mu I)\,(-T)^{-1}\left[A(-T)^{-1}\right]^{s-j}\right)\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T)^{-1}\right]^{j+1}\right)$$
$$+\left[A(-T')^{-1}\right]^{s}\left(\frac{A'}{\mu+\alpha}\right)(\mu,0)(-T)^{-1}\left(\sum_{j=0}^{S-2s-2}\left[A\,(-T')^{-1}\right]^{j}\right)+1 \qquad (17)$$
$$\left.+\sum_{j=0}^{s-1}\left[A\,(-T')^{-1}\right]^{j+1}\right]\underline{e}+\left[A(-T)^{-1}\right]^{s}\frac{A'}{\mu+\alpha}\Bigg\}.$$

Equation 9-17 present all the probabilities of the system.

MODEL-2:-VARYING BULK DEMAND RATES

The following are the assumptions of the model.

- The maximum capacity of the inventory is S and ordering level is s (S-s > s).
- The arrival rate of unit demand is λ.
- The rate of occurrence of bulk demand is α_1. If it does not occur within an exponential time with parameter c the bulk demand arrival rate changes to α_2. Upon occurrence of a demand the rate becomes α_1.
- When the inventory falls to the level s from above an order for S-s units are placed. The lead time distribution for such an order is exponential with parameter μ.
- When a bulk demand occurs an order is placed for S units, the lead time distribution is exponential with rate β and order pending if any is cancelled.

We may find using the method described in Model-1 the time to occurrence of the bulk demand has pdf g(y) and Cdf G(y) as follows.

$$g(y)=\frac{(\alpha_1-\alpha_2)(c+\alpha_1)}{(c+\alpha_1-\alpha_2)}e^{-y(\alpha_1+c)}+\frac{c\alpha_2 e^{-\alpha_2 y}}{(c+\alpha_1-\alpha_2)} \qquad (18)$$

$$G(y)=1-p'e^{-y(\alpha_1+c)}-q'e^{-\alpha_2 y} \qquad (19)$$

Where,

$$p'=\frac{\alpha_1-\alpha_2}{(c+\alpha_1-\alpha_2)}$$

and

$$q'=\frac{c}{(c+\alpha_1-\alpha_2)}$$

We note that $p'+q'=1$.

We also note that the continuous time Markov chain of the model has the state space

$$S=\{(i,j):0\le i\le S,\ j=1,2\}\cup\{0^*\}$$

The inventory is in the state (i, j) when i units are in the inventory and the bulk arrival rate is α_j for $0 \le i \le S$ and j = 1. 2 The inventory is in state 0^* when the lead time rate is β for bulk order of size S. The infinitesimal generator Q of the system can be partitioned as follows:

$$
Q = \begin{array}{c} \\ \underline{S} \\ \underline{S-1} \\ \underline{S-2} \\ \vdots \\ \underline{S-s+1} \\ \underline{S-s} \\ \vdots \\ \underline{s+1} \\ \underline{s} \\ \underline{s-1} \\ \underline{s-2} \\ \vdots \\ \underline{3} \\ \underline{2} \\ \underline{1} \\ 0 \\ 0^* \end{array}
\begin{array}{ccccccccccccccccc}
\underline{S} & \underline{S-1} & \underline{S-2} & \cdots & \underline{S-s+1} & \underline{S-s} & \cdots & \underline{s+1} & \underline{s} & \underline{s-1} & \underline{s-2} & \cdots & \underline{3} & \underline{2} & \underline{1} & 0 & 0^* \\
\hat{T} & \lambda I & & & & & & & & & & & & & & & \underline{\alpha}' \\
0 & \hat{T} & \lambda I & & & & & & & & & & & & & & \underline{\alpha}' \\
 & & \ddots & \ddots & & & & & & & & & & & & & \vdots \\
 & & & \ddots & \ddots & & & & & & & & & & & & \underline{\alpha}' \\
 & & & & \hat{T} & \lambda I & & & & & & & & & & & \underline{\alpha}' \\
 & & & & & \hat{T} & \ddots & & & & & & & & & & \vdots \\
 & & & & & & \ddots & \lambda I & & & & & & & & & \vdots \\
 & & & & & & & \hat{T} & \lambda I & & & & & & & & \underline{\alpha}' \\
\mu I & & & & & & & & \hat{T}' & \lambda I & & & & & & & \underline{\alpha}' \\
 & \mu I & & & & & & & & \hat{T}' & \lambda I & & & & & & \underline{\alpha}' \\
 & & \mu I & & & & & & & & \hat{T}' & \lambda I & & & & & \underline{\alpha}' \\
 & & & & & & & & & & & \hat{T}' & \ddots & & & & \vdots \\
 & & & \ddots & & & & & & & & & \ddots & \ddots & & & \vdots \\
 & & & & & & & & & & & & & \ddots & \ddots & & \vdots \\
 & & & & \mu I & & & & & & & & & & \hat{T}' & \lambda I & \underline{\alpha}' \\
 & & & & & \mu I & & & & & & & & & & \hat{T}_0' & \underline{\alpha}' \\
(\beta, 0) & & & & & & & & & & & & & & & & -\beta
\end{array} \tag{20}
$$

The sub matrices inside the infinitesimal generator Q are given as follows

$$
\hat{T} = \begin{bmatrix} -\lambda - \alpha_1 - c & c \\ 0 & -\lambda - \alpha_2 \end{bmatrix}, \qquad \hat{T}' = \begin{bmatrix} -\lambda - \alpha_1 - c - \mu & c \\ 0 & -\lambda - \alpha_2 - \mu \end{bmatrix}
$$
$$
\hat{T}_o = \begin{bmatrix} -\alpha_1 - c - \mu & c \\ 0 & -\alpha_2 - \mu \end{bmatrix}, \qquad \underline{\alpha}' = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{21}
$$

The matrix Q is of order 2(S + 1) + 1 and all the unmarked entries are zero. Let $\underline{\Pi}$ be the vector of steady state probabilities associated with Q satisfying

$$
\underline{\Pi} Q = 0 \text{ and } \underline{\Pi} \ \underline{e} = 1 \tag{22}
$$

Where:

$$
\underline{\Pi} = \left(\underline{\Pi}_{S,} \underline{\Pi}_{S-1}, \cdots \underline{\Pi}_1, \Pi_0, \Pi_{o*} \right) \cdot
$$

As in the Model-1, we may note that here also

$$
\underline{\Pi}_{s-j} = \underline{\Pi}_{s+1} \left[(\lambda I)(-\hat{T}')^{-1} \right]^{j+1} \text{ for } j = 0,1,2, \ldots, s\text{-}1. \tag{23}
$$

Similarly we get,

$$\Pi_0 = \underline{\Pi}_{s+1}\left[(\lambda I)(-\hat{T}')^{-1}\right]^s \left[(\lambda I)(-\hat{T}_o)^{-1}\right] \tag{24}$$

Using the first column of Q

$$\underline{\Pi}_S \hat{T} + \underline{\Pi}_s \mu I + \Pi_{0^*}(\beta, 0) = 0$$

gives

$$\underline{\Pi}_S = \underline{\Pi}_{s+1}\left[(\lambda I)(-\hat{T}')^{-1}\right](\mu I)(-\hat{T})^{-1} + \Pi_{0^*}(\beta,0)(-\hat{T})^{-1} \tag{25}$$

Proceeding as in model 1, we may note,

$$\underline{\Pi}_{S-i} = \underline{\Pi}_{s+1}\left[\sum_{j=1}^{j=i+1} \left[(\lambda I)(-\hat{T}')^{-1}\right]^j (\mu I)(-\hat{T})^{-1}\left[(\lambda I)(-\hat{T})^{-1}\right]^{i+1-j}\right] + \Pi_{0^*}(\beta,0)(-\hat{T})^{-1}\left[(\lambda I)(-\hat{T})^{-1}\right]^i$$

for $1 \le i \le s$.

We further obtain using similar arguments,

$$\underline{\Pi}_{s+j} = \underline{\Pi}_{s+1}\left[(-\hat{T})(\lambda I)^{-1}\right]^{j-1} \tag{27}$$

for $2 \le j \le S - 2s - 1$.

We use the last column of matrix Q to obtain $\mathbf{\Pi}_0$*.

The equation

$$\underline{\Pi}_S \underline{\alpha}' + \underline{\Pi}_{S-1}\underline{\alpha}' + \cdots + \Pi_o \underline{\alpha}' - \Pi_{o^*}\beta = 0$$

gives

$$\begin{aligned}
&\underline{\Pi}_{s+1}\left\{\sum_{i=0}^{s}\left[\sum_{j=1}^{i+1}\left[(\lambda I)(-\hat{T}')^{-1}\right]^j (\mu I)(-\hat{T})^{-1}\left[(\lambda I)(-\hat{T})^{-1}\right]^{i+1-j}\right]\right\}\underline{\alpha}' \\
&+\Pi_{0^*}(\beta,0)(-\hat{T})^{-1}\left[\sum_{i=0}^{s}\left[(\lambda I)(-\hat{T})^{-1}\right]^i\right]\underline{\alpha}' + \underline{\Pi}_{s+1}\left(\sum_{j=1}^{S-2s-1}\left[(-\hat{T})^{-1}(\lambda I)^{-1}\right]^{j-1}\right)\underline{\alpha}' \\
&+\underline{\Pi}_{s+1}\left(\sum_{j=0}^{s-1}\left[(\lambda I)(-\hat{T}')^{-1}\right]^{j+1}\right)\underline{\alpha}' + \underline{\Pi}_{s+1}\left[(\lambda I)(-\hat{T}')^{-1}\right]^s\left[(\lambda I)(-\hat{T}_o)^{-1}\right]\alpha' - \Pi_{0^*}\beta = 0
\end{aligned}$$

We obtain

$$\begin{aligned}
\Pi_{0^*} = \underline{\Pi}_{s+1}&\left\{\sum_{j=1}^{s}\sum_{j=1}^{i+1}\left[\left[(\lambda I)(-\hat{T}')^{-1}\right]^j (\mu I)(-\hat{T})^{-1}\left[(\lambda I)(-\hat{T}')^{-1}\right]^{i+1-j}\underline{\alpha}'\right.\right. \\
&+\sum_{j=1}^{S-2s-1}\left[(-\hat{T})(\lambda I)^{-1}\right]^{j-1}\underline{\alpha}' + \sum_{j=0}^{s-1}\left[(\lambda I)(-\hat{T}')^{-1}\right]^{j+1}\underline{\alpha}' \\
&\left.+\left[(\lambda I)(-\hat{T}')^{-1}\right]^s\underline{\left[(\lambda I)(-\hat{T}_o)^{-1}\right]}\alpha'\right\} \\
&\left\{\beta - (\beta,0)(-\hat{T})^{-1}\sum_{i=0}^{s}\left[(\lambda I)(-\hat{T})^{-1}\right]^i\underline{\alpha}'\right\}^{-1}
\end{aligned} \tag{28}$$

Using total law of probability $\underline{\Pi}_{s+1}$ may be calculated. The equation $\underline{\Pi}_S \underline{e} + \underline{\Pi}_{S-1} \underline{e} + \cdots + \underline{\Pi}_1 \underline{e} + \underline{\Pi}_0 \underline{e} + \Pi_{0^*} = 1$ using Eq. 25-28

$$\underline{\Pi}_{s+1} . \underline{a} = 1 \tag{29}$$

and

$$\underline{\Pi}_{s+1} = \frac{\underline{a}^t}{|\underline{a}|^2} \tag{30}$$

$$\begin{aligned}
\underline{a} = & \left\{ \sum_{i=0}^{s} \left[\sum_{j=1}^{i+1} \left[(\lambda I)(-\hat{T}')^{-1} \right]^j (\mu I)(-\hat{T})^{-1} \left[(\lambda I)(-\hat{T})^{-1} \right]^{i+1-j} \right] \right\} \underline{e} \\
& + \left(\sum_{j=1}^{S-2s-1} \left[(-\hat{T})^{-1} (\lambda I)^{-1} \right]^{j-1} \right) \underline{e} + \left[\sum_{j=0}^{S-2s-1} \left[(-\hat{T})(\lambda I)^{-1} \right]^{j+1} \right] \underline{e} \\
& + \left[(\lambda I)(-\hat{T}')^{-1} \right]^s \left[(\lambda I)(-\hat{T}_0)^{-1} \right] \underline{e} \\
& + \left\{ \sum_{i=0}^{s} \sum_{j=1}^{i+1} \left[(\lambda I)(-\hat{T}')^{-1} \right]^j (\mu I)(-\hat{T})^{-1} \left[(\lambda I)(-\hat{T})^{-1} \right]^{i+1-j} \right\} \underline{\alpha}' \\
& + \left(\sum_{j=1}^{S-2s-1} \left[(-\hat{T})(\lambda I)^{-1} \right]^{j-1} \right) \underline{\alpha}' + \left[\sum_{j=0}^{s-1} \left[(\lambda I)(-\hat{T}')^{-1} \right]^{j+1} \right] \underline{\alpha}' \\
& + \left[(\lambda I)(-\hat{T}')^{-1} \right]^s \left[(\lambda I)(-\hat{T}_0)^{-1} \right] \underline{\alpha}' \Big\} \\
& \left\{ \beta - (\beta, 0)(-\hat{T})^{-1} \sum_{i=0}^{s} \left[(\lambda I) \ (-\hat{T})^{-1} \right]^i \underline{\alpha}' \right\}^{-1} \\
& \left\{ 1 + (\beta, 0)(-\hat{T})^{-1} \left\{ \sum_{i=0}^{s} \left[(\lambda I) \ (-\hat{T})^{-1} \right]^i \right\} \underline{e} \right\}
\end{aligned} \tag{31}$$

Equation 23-30 present all the probabilities of the system.

NUMERICAL EXAMPLE

Numerical example for model 1: Let the maximum capacity of the inventory be 5(S = 5) and the reorder level is 2(s = 2). The infinitesimal generator order (12) of the finite state space continuous time Markov chain is as follows.

$$\begin{array}{c} \\ \underline{5} \\ \underline{4} \\ \underline{3} \\ \underline{2} \\ \underline{1} \\ 0 \\ 0^* \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \underline{5} & \underline{4} & \underline{3} & \underline{2} & \underline{1} & 0 & 0^* \end{array} \\
\left[\begin{array}{ccccccc}
T & A & & & & & \underline{\alpha}' \\
 & T & A & & & & \underline{\alpha}' \\
 & & T & A & & & \underline{\alpha}' \\
\mu I & & & T' & A & & \underline{\alpha}' \\
 & \mu I & & & T' & A' & \underline{\alpha}' \\
 & & (\mu, 0) & & & -\mu-\alpha & \alpha \\
(\beta, 0) & & & & & & -\beta
\end{array} \right]
\end{array}$$

where, T, A, T′ A ′ $\underline{\alpha}'$ are as given:

$$T = \begin{bmatrix} -\lambda_1 - \alpha - c & c \\ 0 & -\lambda_2 - \alpha \end{bmatrix},$$

$$T' = \begin{bmatrix} --\lambda_1 - \alpha - c - \mu & c \\ 0 & -\lambda_2 - \alpha - \mu \end{bmatrix},$$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ \lambda_2 & 0 \end{bmatrix}, A' = \begin{bmatrix} \lambda_1 \\ \lambda 2 \end{bmatrix}, \underline{\alpha}' = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}.$$

For fixed values of $\lambda_1 = 1$, $\lambda_2 = 2$, $\alpha = 1$, $C = 1$, $\mu = 2$ and $\beta = 2$, we compute the components of $\underline{\Pi}$. The steady state probability vector

$$\underline{\Pi} = \left(\Pi_5, \Pi_4, \Pi_3, \Pi_2, \Pi_1, \Pi_0, \Pi_{0^*} \right)$$

is given by $\Pi_5 = 0.3240$, $\Pi_4 = 0.1890$, $\Pi_3 = 0.1085$, $\Pi_2 = 0.0325$, $\Pi_1 = 0.0091$, $\Pi_0 = 0.0035$, $\Pi_0^* = 0.3333$. The sum of steady state probabilities is found to be 1.0000.

Table 1: Compute the e.i.l for λ_1 = 1, 2, 3, 4 and 5

λ_1	e.i.l
1	2.7756
2	2.5629
3	2.3785
4	2.2141
5	2.0666

Table 2: Compute the e.i.l for c = 1, 2, 3, 4 and 5

C	e.i.l
1	2.7756
2	2.7299
3	2.7005
4	2.6799
5	2.6647

Table 3: Compute the e.i.l for λ_2 = 1, 2, 3, 4 and 5

λ_2	e.i.l
1	2.8569
2	2.7756
3	2.7299
4	2.7005
5	2.6799

Case 1: (Varying first demand rate λ_1) For the fixed values of c = 1 , λ = 2, μ = 2, α = 1 and β = 2 we compute the e.i.l for λ_1 = 1, 2, 3, 4 and 5. They are given in Table 1.

It is clear from Table 1, when the first demand rates λ_1 increase, the e.i.l decrease.

Case 2: (Varying c) For the fixed values of λ_1 = 1 , λ_2 = 2, μ = 2, α = 1 and β = 2 we compute the e.i.l for c = 1, 2, 3, 4 and 5. They are given in Table 2.

It is clear from Table 2, when the first demand rates c increase, the e.i.l decrease.

Case 3: (Varying second demand rate λ_2) For the fixed values of λ_1 = 1 , c= 2, μ = 2, α = 1 and β = 2 we compute the e.i.l for λ_2 = 1, 2, 3, 4 and 5. They are given in Table 3.

It is clear from Table 3, when the second demand rates increase, the e.i.l decrease.

Case 4: (Varying lead time μ when calamity occurs) For the fixed values of λ_1 = 1 , c= 1, λ_1 = 2, α = 1 and β = 2 we compute the e.i.l for μ = 1, 2, 3, 4 and 5. They are given in Table 4.

It is clear from Table 4, when the lead time μ (for calamity order) increase, the e.i.l increase.

Case 5: (Increasing calamity rate α) For the fixed values of λ_1 = 1, c= 1,λ_2= 2, α= 2 and β = 2 we compute the e.i.l for α = 1, 2, 3, 4 and 5. They are given in Table 5.

It is clear from Table 5, when calamity occurrence rate increase, the e.i.l decrease.

Case 6: (Varying lead time rate when calamity occurs) For the fixed values of λ_1 = 1, c = 1, λ_2 = 2, α = 1 and μ = 2 we compute the e.i.l for β = 1, 2, 3, 4 and 5. They are given in Table 6.

It is clear from Table 6, when the lead time β (for calamity order) increase, the e.i.l increase.

Table 4: Compute the e.i.l for μ = 1, 2, 3, 4 and 5

μ	e.i.l
1	2.7092
2	2.7756
3	2.81
4	2.8308
5	2.8447

Table 5: Compute the e.i.l for α = 1, 2, 3, 4 and 5

α	e.i.l
1	2.7756
2	2.2418
3	1.8552
4	1.5751
5	1.3658

Table 6: Compute the e.i.l for β = 1, 2, 3, 4 and 5

β	e.i.l
1	2.0817
2	2.7756
3	3.1226
4	3.3307
5	3.4695

Numerical example for model-2: Let the maximum capacity of the inventory be 5 (S = 5) and the reorder level is 2 (s = 2). The infinitesimal generator order (13) of the finite state space continuous time Markov chain is as follows:

$$\begin{array}{c} \\ \underline{5} \\ \underline{4} \\ \underline{3} \\ \underline{2} \\ \underline{1} \\ 0 \\ 0^* \end{array} \begin{array}{c} \begin{array}{ccccccc} \underline{5} & \underline{4} & \underline{3} & \underline{2} & \underline{1} & 0 & 0^* \end{array} \\ \begin{bmatrix} \hat{T} & \lambda I & & & & & \underline{\alpha}' \\ & \hat{T} & \lambda I & & & & \underline{\alpha}' \\ & & \hat{T} & \lambda I & & & \underline{\alpha}' \\ \mu I & & & \hat{T}' & \lambda I & & \underline{\alpha}' \\ & \mu I & & & \hat{T}' & \lambda I & \underline{\alpha}' \\ & & \mu I & & & \hat{T}_0 & \alpha \\ (\beta, 0) & & & & & & -\beta \end{bmatrix} \end{array}$$

where, T, A, T′ A′ $\underline{\alpha}'$ are as given:

$$\hat{T} = \begin{bmatrix} -\lambda - \alpha_1 - c & c \\ 0 & -\lambda - \alpha_2 \end{bmatrix},$$

$$\hat{T}' = \begin{bmatrix} -\lambda - \alpha_1 - c - \mu & c \\ 0 & -\lambda - \alpha_2 - \mu \end{bmatrix},$$

$$\hat{t}_0 = \begin{bmatrix} \alpha_1 - c - \mu & c \\ 0 & -\alpha_2 - \mu \end{bmatrix},$$

$$\underline{\alpha}' = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

Table 7: Compute the e.i.l for α_1 = 1, 2, 3, 4 and 5

α_1	e.i.l
1	2.5678
2	2.2788
3	2.0439
4	1.8509
5	1.6901

Table 8: Compute the e.i.l for c = 1, 2, 3, 4 and 5

C	e.i.l
1	2.6643
2	2.5678
3	2.5098
4	2.4711
5	2.4434

Table 9: Compute the e.i.l for α_2 = 1, 2, 3, 4 and 5

α_2	e.i.l
1	2.8569
2	2.6643
3	2.5678
4	2.5098
5	2.4711

For fixed values of $\alpha_1 = 1, \alpha_2 = 2$, $\lambda = 1$, $\mu = 2$ and $\beta = 2$, we compute the components of $\underline{\Pi}$. The steady state probability vector

$$\underline{\Pi} = \left(\Pi_{5,}\Pi_{4,}\Pi_{3},\Pi_{2,}\Pi_{1,}\Pi_{0,}\Pi_{0^*}\right)$$

is given by Π_5 = 0.3660, Π_0 = 0.1543, Π_3 = 0.0622, Π_2 = 0.0137, Π_1 = 0.0030, Π_0 = 0.0008, Π_0^* = 0.4000. The sum of steady state probabilities is found to be 1.0000.

Case 1: (e.i.l for the increased first arrival rate α_1 (bulk demand)). For the fixed values of c = 2 α_2 = 2, μ = 2, λ= 1 and β = 2 we compute the e.i.l for α_1 = 1, 2, 3, 4 and 5. They are given in Table 7.

It is clear from Table 7, when the bulk demand rates increase, the e.i.l decrease.

Case 2: For the fixed values of α_1 = 1, α_2 = 2, μ = 2, λ= 1 and μ = 2, we compute the e.i.l for c = 1, 2, 3, 4 and 5. They are given in Table 8.

It is clear from Table 8, when the first demand rates c increase, the e.i.l decrease.

Case 3: (e.i.l for the increased second arrival rate α_2 (bulk demand)) For the fixed values of α_1 = 1, c = 1, μ = 2, λ= 1 and β = 2 , we compute the e.i.l for α_2 = 1, 2, 3, 4 and 5. They are given in Table 9.

It is clear from Table 9, when the second demand rates increase, the e.i.l decrease.

Case 4: (e.i.l for the increased lead time μ (unit demand)) For the fixed values of α_1 = 1, c = 1,α_2 = 2, λ = 1 and β = 2, we compute the e.i.l for μ = 1, 2, 3, 4 and 5. They are given in Table 10.

Table 10: Compute the e.i.l for μ = 1, 2, 3, 4 and 5

μ	e.i.l
1	2.6445
2	2.6643
3	2.6758
4	2.6832
5	2.6883

Table 11: Compute the e.i.l for λ = 1, 2, 3, 4 and 5

λ	e.i.l
1	2.6643
2	2.4245
3	2.2238
4	2.0473
5	1.8907

Table 12: Compute the e.i.l for β = 1, 2, 3, 4 and 5

β	e.i.l
1	1.9031
2	2.6643
3	3.0742
4	3.3304
5	3.5057

It is clear from Table 10, when the lead time μ(when unit demand occurs) increase, the e.i.l increase.

Case 5: (e.i.l for the increased demand rate λ (unit demand)) For the fixed values of α_1 = 1, c = 1,α_2 = 2, μ = 2 and β = 2, we compute the e.i.l for λ = 1, 2, 3, 4 and 5. They are given in Table 11.

It is clear from Table 11, when calamity occurrence rate increase, the e.i.l decrease.

Case 6: (e.i.l for the increased lead time β (bulk demand)) For the fixed values of α_1 = 1, c = 1, α_2 = 2, μ = 2 and λ = 1, we compute the e.i.l for β = 1, 2, 3, 4 and 5. They are given in Table 12.

It is clear from Table 12, when lead time (when bulk demand occurs) increase, the e.i.l also increase.

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